Colouring Oriented Graphs on Surfaces

by

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Abstract

This thesis considers the existence of certain vertex and edge colourings of oriented graphs G, primarily oriented colourings, given the topological structure of G. The main measures of the topology of G we consider is the Euler genus g, maximum degree Δ, and degeneracy d. In doing so we present results by the author appearing in [18, 21], particularly as they pertain to the oriented chromatic number of a graph G, denoted $\chi_o(G)$. The first result is that for all $k \geq 2$ and $d \geq \log_2(k)$, if G is a d-degenerate graph with $\chi_2(G) \leq k$ (2-dipath chromatic number), then $\chi_o(G) \leq \frac{33}{10}kd^2 2^d$, which greatly improves the prior bound of the form $\chi_o(G) \leq 2^k - 1$ from MacGillivray, Raspaud, and Swartz [48] whenever $\log_2(k) \leq d \ll k$. Additionally we give constant factor asymptotic improvements on bounds for $\chi_o(G)$ in terms of maximum degree and degeneracy from Kostochka, Sopena, and Zhu [46] as well as Aravind and Subramanian [8]. Our final and largest contribution is to prove that the oriented chromatic number is at most g^{6400} for all graphs with Euler genus at most g, improving the prior asymptotic upper bound $\chi_o(G) \leq 2^{O(g^{1/2+\varepsilon})}$ shown by Aravind and Subramanian [8].

Keywords: graph theory; oriented colouring; 2-dipath colouring; surfaces; probabilistic method

Dedication

This thesis is dedicated to my friends and family who have supported me through this degree. Your help is invaluable.

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I would like to take this opportunity to acknowledge the many people who directly and indirectly made my mathematical journey so far possible.

I am grateful to my family for their constant support. Although we have not been able to see each other nearly as much I would like over the past two years, it would be impossible for me to be where I am without you. I miss you tremendously and I promise to continue doing everything I can to make this time apart worth it.

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Chapter 1

Aims of the Thesis

An *oriented graph* G is a directed graph whose underlying graph is simple. Throughout this thesis more often than not the graphs we consider are oriented graphs, so it should be understood that stating G is a graph is synonymous with stating G is an oriented graph, unless otherwise stated. We identify parameters of the underlying graph of an oriented graph G as would be normally done given simple graphs and parameters of the orientation of G as is standard with directed graphs.

For example $deg(v)$ is the degree of the vertex v independent of orientation whereas $deg^+(v)$ denotes the out-degree, and $deg^-(v)$ the in-degree of the vertex v. Similarly, a path P in the graph need not be a directed path. When a path is directed we will call it a *dipath*. Additionally, given an oriented graph G and a parameter of simple graphs, such as chromatic number χ, we let the parameter of G, in this example $\chi(G)$, be the parameter of the underlying simple graph of G, in this case the chromatic number of the underlying simple graph of G.

The goal of this thesis it to present work by the author from [18] and [21] concerning graph colouring in several classes of sparse graphs. We will focus on oriented colouring graphs $G = (V, E)$ with at most $O(|V|)$ numbers of edges. The thesis is structured as follows.

In Chapter 2, we introduces elements of the graph theory literature as they pertain to the rest of the thesis. This includes a short survey of graphs on surfaces, as colouring graphs on surfaces is a major part of the thesis. We give an outline of the probabilistic method, which will serve as a primary tool in our proofs later in the thesis. We addition give a survey of the literature concerning graph colouring variants, including but not limited to oriented colouring. The colouring parameters we consider here are notable due to their significance to oriented colouring, in particular, the oriented colouring results which make up the novel contributions of the author presented in this thesis. Basic familiarity with graph theory is assumed. For readers unfamiliar with graph theory we recommend [83] as a reference.

Chapter 3 is devoted to studying graphs with bounded maximum degree and degeneracy.Here we will present work from [21] and [46], which established the current best upper and lower bounds, respectively, for the oriented chromatic number in terms of maximum degree. Along with this, Chapter 3 introduces a novel method for bounding the oriented chromatic number in terms of degeneracy and 2-dipath chromatic number. This technique will be useful again in Chapter 4.

In Chapter 4 we consider the oriented chromatic number of graphs on surfaces. The focus of this is to present work from [18] which shows that the oriented chromatic number is bounded above by a polynomial in Euler genus. This resolves a conjecture of Aravind and Subramanian from [8]. We additionally apply methods presented in Chapter 3 from [21] to show that the best upper bound for the oriented chromatic number of a surface of large Euler genus cannot be much larger than the best upper bound for the 2-dipath chromatic number of the same surface. This provides a method to improve our upper bound on the oriented chromatic number in terms of Euler genus.

We conclude in Chapter 5 by summarising the problems which remain open for future work. As part of this, we give a list of conjectures.

Chapter 2

Preliminaries

2.1 Planar Graphs

Given planar graphs are normally one of the first topics covered in a introduction to graph theory course, this means we will not devote a large amount of time to them here. All the same, given this is a thesis we introduce the major results in the area, including in this case planar graphs.

A simple graph $G = (V, E)$ is *planar* if it can be embedded (i.e. drawn) in the plane without edge crossings. If G is a planar graph, we call a drawing of G in the plane without edge crossings, a *plane embedding*. Throughout this thesis, embeddings into the plane, and onto a more general surface, are normally denoted by Π.

Planar graphs are some of the oldest and most widely studied graphs. For example, many problems in geometry and other areas correspond to problems in graph theory. For one example of this we draw attention to the fact that the skeletons of convex polyhedra are in bijection with 3-connected planar graphs (see Steinitz's theorem in [80]). Moreover, the study of planar graphs originates from the famous 4-colour problem proposed in 1852 by Guthrie, which serves as the origins for many problems in graph theory.

One of the most important results regarding planar graphs traces its way back to Euler some time around 1740. This might be surprising to some readers given we have just stated planar graphs were first studied in the mid 19 century. Such readers would be correct to questions this. We note that Euler proved his formula, stated below, originally for polyhedra. The statement of Euler's formula for planar graphs we list below is credited to Euler due to the already mentioned relationship between planar graphs and convex polyhedra.

Theorem 2.1.1 (Euler's formula for planar graphs)**.** *Let $G = (V, E)$ be a connected graph and* Π *a fixed plane embedding of G. Letting F be the set of faces in the embedding* Π*,*

$$|V| - |E| + |F| = 2.$$

Next, we note a significant characterisation of planar graphs, which was proved by Wagner [82]. We note, that Wagner's theorem is equivalent to another well known characterisation of planar graphs provided by Kuratowski in [47]. Significantly, Kuratowski published his theorem 7 years before Wagner, and Wagner's proof is based on Kuratowski's result. We include Wagner's theorem because it's forbidden minor characterisation of planar graphs fits slightly better with Hadwiger's conjecture and the work of Robertson and Seymour on graph families characterised by forbidden minors which we will see later.

Theorem 2.1.2 (Wagner's theorem [82])**.** *Let G be a graph. Then G is planar if and only if G contains neither K_5 nor $K_{3,3}$ as a minor.*

Of course, the most celebrated theorem about planar graphs, and perhaps in all of graph theory, is the 4-colour theorem. We assume the reader is familiar with proper vertex colouring and the chromatic number $\chi(G)$ of a graph G. For completeness, and as the 4-colour theorem is the root of a problem we study in this thesis, we state the 4-colour theorem below.

Theorem 2.1.3 (4-colour theorem [5, 6, 7])**.** *If G is a planar graph, then $\chi(G) \leq 4$.*

As this thesis studies a generalisation of the 4-colour problem in Chapter 4, we conclude this subsection by mentioning the most famous generalisation of the 4-colour problem. This is of course Hadwiger's conjecture, which can be viewed as leveraging Wagner's theorem to find a topology-free version of the 4-colour problem.

Conjecture 2.1.4 (Hadwiger's conjecture)**.** *If G is a graph with no K_n minor, then $\chi(G) \leq n - 1$.*

2.2 Graphs on Other Surfaces

One of the most natural ways to generalise the notion of planar graphs besides considering graphs with forbidden minors is to pick a surface S that is not the plane, and ask, which graphs can be embedded (i.e. drawn) on S without edge crossings? For convenience later, unless otherwise stated,

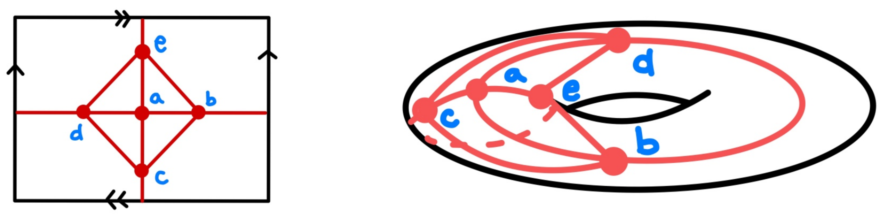

Figure 2.1: See two representations of a drawing of K_5 embedded on a torus.

we assume all embeddings we discuss have no edge crossings. Here an embedding Π of a graph G on a surface S is a 2-cell embedding if and only if every face of Π is homeomorphic to a disc. Colloquially, this assumption is required to ensure that if we are discussing an embedding of G on a surface S, then all the topology of S is "used" by our embedding. As a general reference for graphs on surfaces we recommend [58].

We begin by noticing that for every graph G, there exists a surface S such that G has a 2-cell embedding on S. This is because we can embed G in the plane, potentially with many edge crossings, then for each edge crossing we can add a handle to the plane to generate a surface S that G embeds onto without edge crossings. If such an embedding is not a 2-cell embedding, then we can create a simpler surface which G has a 2-cell embedding onto. For example, consider Figure 2.1 to see how K_5 can be embedded onto a torus, while we have already seen that K_5 cannot be embedded in the plane.

Given that every graph can be embedded onto some surface, it is natural to measure the complexity of a graph by the surfaces it can embed onto. In particular, we extend the notion of genus and Euler genus from surfaces to graphs as follows. Given a graph G we let the *Euler genus* of G, $g(G)$, or simply g when the choice of G is clear from context, be the least integer g such that G embeds without edge crossings on an orientable surface with $2g$ handles, or a non-orientable surface with g cross-caps. Given this definition of the Euler genus of a graph, we note that Euler's formula for planar graphs, see Theorem 2.1.1, extends to graphs on other surfaces as follows.

Theorem 2.2.1 (Euler's formula)**.** *Let $G = (V, E)$ be a connected graph, S a surface of Euler genus g, and Π a fixed 2-cell embedding of G on S. Letting F be the set of faces in the embedding Π,*

$$|V| - |E| + |F| = 2 - g.$$

We note that considering graphs embedded on more general surfaces has a long history of being studied in the literature. The example of which the author is aware of being by Heawood in 1890, who asked if the the chromatic number of a graph can be bounded as a function of its Euler genus. The answer is yes. As one might imagine of a proof that is almost 140 years old, Heawood's proof proceeds by a relatively simple argument, not unlike the 6-colour theorem. Headwood uses Euler's formula to bound the degeneracy of a graph with Euler genus at most g, by a function that is $O(\sqrt{g})$, which implies that that there is a $O(\sqrt{g})$-colouring of G that can be generated using a greedy argument.

Theorem 2.2.2 (Heawood's map colouring theorem [37])**.** *If G is a graph of Euler genus g, then $\chi(G) = O(\sqrt{g})$*

We note that Heawood's approach does little to indicate if this upper bound is asymptotically correct. Moreover, at the time there was no known way of constructing graphs with high chromatic number and low Euler genus. This lead Heawood to conjecture that his bound was tight, a problem that became known as the Heawood map colouring problem.

In fact, Heawood was correct. However, the process of proving this was highly non-trivial. It did take approximately 85 years for Heawood's problem to be fully resolved. Finally, after great effort, the problem was fully resolved by work from Ringel and Youngs. Heawood's problem was resolved by calculating the Euler genus of all complete graphs. The Euler genus of each complete graph is listed below.

Theorem 2.2.3 ([70, 69])**.** *Let $n \geq 3$ be an integer. Then the complete graph K_n has Euler genus*

$$\left\lceil \frac{(n-3)(n-4)}{6} \right\rceil .$$

Of course, $\chi(K_n) = n$ implying that if $g(K_n) = \Omega(n^2)$, then there exists a graph with Euler genus g and chromatic number on the order of $\Omega(\sqrt{g})$. Hence, the Ringel-Youngs' theorem implies Heawood's map colour conjecture is correct.

Corollary 2.2.4. *There exists a graph G of Euler genus g, and $\chi(G) \geq \Omega(\sqrt{g})$.*

Another significant example of how graphs on surfaces are studied in the literature is in their relationship to graph minor theory. As we have seen with Wagner's theorem, at least for planar graphs, the structure of graph embeddings and forbidden minors are closely related. It turns out that this relationship is not a coincidence resulting from some special property of the plane, but rather the first instance of a much more general pattern.

Notice that for any surface S, the graphs that can be embedded on S without edge crossings form a minor closed class. Here a minor closed class is a family of graphs $\mathcal{G}$ such that if $G \in \mathcal{G}$ and H is a minor of G, then $H \in \mathcal{G}$. Thus, if G can be embedded on S without edge crossings and H is a minor of G, then H can be embedded on S without edge crossings. Given this, a celebrated result of Robertson and Seymour, often dubbed the Robertson-Seymour Theorem, which is stated below, implies that for all S there is a finite list of graph $H_1, \ldots, H_k$ such that G can be embedded on S if and only if G does not contain H_i as a minor for any i.

Theorem 2.2.5 (Robertson-Seymour Theorem [72])**.** *Let $\mathcal{G}$ be a family of graphs closed on minors. Then, there exists a finite list of graph $H_1, \ldots, H_k$ such that $G \in \mathcal{G}$ if and only if G does not contain H_i as a minor for all $1 \leq i \leq k$.*

The proof of the Robertson-Seymour theorem is not constructive, in the sense that it does not provide means to take a minor closed family of graphs and provide the list of forbidden minors that characterise the family. Of course, this means that Wagner's theorem is not directly implied by the Robertson-Seymour theorem. Moreover, the list of forbidden minors which characterise the graphs embeddable on a surface S are unknown for almost every surface S. Given a minor closed family of graphs, we say that the forbidden minors which characterise this family are *obstructions*.

Even the set of obstructions for the torus remains unknown. We note here that Myrvold and Woodcock [60] summarise and verify previously released, but not peer-reviewed, work by [20, 40, 41, 63, 64, 85] to show that the shortest list of obstructions for the torus contains at least 17523 graphs. The methods from these sources are diverse but are largely computational. As a result they do not shed much light on if this list is close to being complete. Moreover, the graphs that make up this list are extremely diverse, and it is even unclear if there exist entire classes of obstructions for the torus that are yet unknown.

It is important to note that the relationship between classes of graphs which are characterised by a set of forbidden minors, and graphs on surfaces runs deeper still. By this we mean that it is not simply the case that the graphs embeddable on a surface are characterised by a list of forbidden minors, as the Robertson-Seymour theorem implies, but it is also true that every minor closed family of graphs is a class of graphs "almost" embeddable on a surface. The word almost here is hiding a lot of details which are not significant to the contents of this thesis. As a result, we will not fully explore this idea here. However, to provide the reader with a proper sense of the literature, we will state the most significant of these results, this being the graph structure theorem. This theorem is also called the excluded minor theorem. The version of the theorem we present here appears in a survey by Kawarabayashi and Mohar [42].

We note that before stating the theorem we must define a vortex. A society in a graph $G = (V, E)$ is a vertex set $F \subset V$ together with a fixed cyclic ordering of F. For $a, b \in F$ for a society F in G, $F(a, b)$ denotes all vertices between a and b in the ordering, but not a or b, while $F[a, b]$ denotes $F(a, b) \cup \{a, b\}$. A vortex is the pair (G, F) where F is a society in G. A vortex has adhesion at most k if for any $a, b \in F$, there are no k-vertex disjoint paths joining $F[a, b]$ and $F(b, a)$.

Let (G, F) be a vortex and k an integer, which satisfy that for all $v \in F$, there exists a subgraph G_v of G such that

1. $v \in V(G_v)$ for all $v \in F$, and
2. the subgraphs G_v are mutually edge disjoint and their union is G, and
3. for all $u, v \in F$, $V(G_v) \cap V(G_u)$ are contained in $\bigcap_{w \in F[u,v]} V(G_w)$ or $\bigcap_{w \in F[v,u]} V(G_w)$, and
4. if $u, v \in F$ and $u \neq v$, then $|V(G_u) \cap V(G_v)| \leq k$.

Then we say that $\{G_v : v \in F\}$ is a vortex decomposition with width $k - 1$ if $k = \max_{v \in F} |V(G_v)|$. The width of a vortex is the minimum t such that (G, F) has a vortex decomposition of width t. We note that Robert and Seymour proved in [74] that if (G, F) is a vortex with adhesion $k + 1$, then (G, F) has width at most k.

Theorem 2.2.6 (Graph Structure Theorem [71])**.** *Let H be a graph. Then, there exits a positive integer k such that every graph G can be constructed as a sequence of clique sums of order at most k of graphs $G^{(1)}, \ldots, G^{(t)}$, defined as follows; let S be a surface on which H does not embed and let $1 \leq r \leq t$ be fixed but arbitrary, $G^{(r)}$ has a set of at most k vertices, A, such that $G^{(r)} - A$ can be written as the union of graphs $G_0^{(r)}, \ldots, G_k^{(r)}$ such that*

1. *$G_0^{(r)}$ is embedded in S, and*
2. *For all $i, j > 0$, $G_i^{(r)}$ and $G_j^{(r)}$ are pairwise disjoint, and*

3. *There are not necessarily distinct faces $F_1, \ldots, F_k$ of $G_0^{(r)}$ in S, and there are pairwise disjoint disks $D_1, \ldots, D_k$ in S, such that for $1 \leq i \leq k$, D_i is a subset of the closure of F_i, and if $U_i = D_i \cap G_0^{(r)} \subset V(G_0^{(r)})$ is cyclically ordered as imposed by the boundary of the disk D_i, then $(G_i^{(r)}, U_i)$ is a vortex of width at most k.*

Clearly, this theorem is both impressive and daunting in its scope and complexity. As we have pointed out the content of the thesis does not require the graph structure theorem, so we will not explore it further. Before moving onto other topics, we mention one more class of theorems regarding the structure of graphs with excluded minors that has been the focus of some effort recently.

These theorems are often called product structure theorems, and concern describing each graph with no H-minor as a subgraph of a graph product, in which each factor cannot be too complicated. We present one such theorem here. Given our focus on graphs on surfaces, we state a product structure theorem related to graphs of bounded Euler genus.

Before stating the theorem we must define the strong product of two graphs. Let G and H be graphs. Then the strong product of G and H, denoted $G \boxtimes H$ is the graph with vertex set $V(G \boxtimes H) = V(G) \times V(H)$ and edge set defined by $((u,x),(v,y)) \in E(G \boxtimes H)$ if and only if $(u,v) \in E(G)$ or $(x,y) \in E(H)$. Notice that the symbol $\boxtimes$ is used to denote the strong product of two graphs because $\boxtimes$ is a picture of the strong product of P_2 with itself. This motif of denoting graph products by the structure of P_2 times P_2 is also used for some other graph products.

Theorem 2.2.7 ([24]). *Every graph with Euler genus g is a subgraph of $H \boxtimes P \boxtimes K_{\max\{2g,3\}}$ for a planar graph H with treewidth 3 and a path P.*

This concludes our literature review for graphs on surfaces. If a reader is still interested we once again recommend [58] for a general reference, the survey of Kawarabayashi and Mohar [42] for a solid review of the connection between graphs on surfaces and excluded minors.

2.3 The Probabilistic Method

One of the primary tools used in the proofs of this thesis is the probabilistic method. As a result, we describe the method briefly here. Those readers already familiar with the method can skip this section.

Like induction, contradiction, or many of the other tool in a working mathematician's tool belt, the probabilistic method is a general strategy for proving statements that are too hard to show by an explicit construction. In particular, the probabilistic method is clever little wrench that lets one to avoid constructing an object, which might otherwise be hard to demonstrate, by indirectly showing that it must exist.

Broadly speaking, the probabilistic method consists of the following strategy: We aim to prove the existence of an object G in a class $\mathcal{G}$ such that G has a special property $\mathcal{P}$. We proceed as follows:

1. Construct a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that Ω is a finite set of objects of class $\mathcal{G}$.
2. Let H be a randomly selected object from Ω.
3. Demonstrate that $\mathbb{P}(H \text{ has property } \mathcal{P}) > 0$.
4. There must be some object $H \in \Omega$ with property $\mathcal{P}$.
5. Let $G = H$ where H is such an object.

As should be immediately clear there are a variety of ways each of these steps might be conducted. For instance, choosing the correct probability space to work in is often more of an art then a science. While even if the choice of space is clear, determining $\mathbb{P}(H \text{ has property } \mathcal{P}) > 0$ may require incredibly high powered techniques from probability, analysis, or some other areas of science. For an example of this see a recent paper by Mattheus and Verstraete [54] which has made a breakthrough in understanding the Ramsey Number $R(4, k)$ through employing ideas from information theory in concert with the probabilistic method.

For readers interested in learning more about this topic or in trying their hand at some interesting, informative, and challenging problems we refer you back to [4, 79]. Otherwise, the probabilistic method will be used to demonstrate work by the author as early as Chapter 3, specifically, in Subsection 3.2 and Subsection 3.3 and Subsection 3.5.

Next, we give some bounds on probabilities that are often useful when conducting the probabilistic method.

Theorem 2.3.1 (Markov's Inequality)**.** *If X is a non-negative random variable and $x > 0$, then*

$$\mathbb{P}(X \geq x) \leq \frac{\mathbb{E}(X)}{x}.$$

Theorem 2.3.2 (Chebyshev's Inequality)**.** *Let X be a random variable with finite variance σ^2 and mean μ. Then for all real numbers $x > 0$,*

$$\mathbb{P}(|X - \mu| \geq x\sigma) \leq \frac{1}{x^2}.$$

Meanwhile the next useful inequality is representative of a wider family of inequalities called Chernoff bounds. We provide a single version of Chernoff bound here, as an example. For those curious, generally speaking Chernoff bounds are shown by bounding the function of t, $\mathbb{E}(e^{tX})$ where X is a random variable we wish to study.

Theorem 2.3.3 (Chernoff's Bound)**.** *Suppose S_n is a random variable with the binomial distribution $Bin(n,p)$ and expectation $\mathbb{E}(S_n) = np = \mu$. Then for $\varepsilon \leq \frac{3}{2}$ we have*

$$\mathbb{P}(|S_n - \mu| \geq \varepsilon\mu) \leq 2\exp\Big(-\frac{\varepsilon^2\mu}{3}\Big).$$

2.4 Acyclic & Forbidden Subgraph Colouring

Given an undirected graph $G = (V, E)$, a proper vertex colouring $\phi : V(G) \to \mathbb{N}$ is acyclic if given any $i, j \in \mathbb{N}$, $G_{i,j} := G[\phi^{-1}(i) \cup \phi^{-1}(j)]$ is acyclic (equivalently a forest). We say ϕ is an *acyclic k-colouring* if the range of ϕ is of cardinality k. Then, the *acyclic chromatic number* of a simple graph G, denoted $\chi_a(G)$, is the least integer k such that G admits an acyclic k-colouring. For an example of an acyclic colouring and a proper colouring which is not acyclic see Figure 2.2.

Since being introduced by Grünbaum [35], the acyclic chromatic number has been extensively studied. For example, Borodin [12] showed that every planar graph is acyclically 5 colourable, which is best possible, Alon, Mohar, and Sanders [3] proved that every graph of Euler genus g has an acyclic $O(g^{4/7})$-colouring, which is tight up to a logarithmic factor. Furthermore, Alon, Mcdiarmid, and Reed [2] proved that every graph with maximum degree Δ has an acyclic $O(\Delta^{4/3})$-colouring, which is again close to tight. We note that this upper bound for acyclic chromatic number in maximum degree has recently been improved by Gonçalves, Montassier, and Pinlou [33] using entropy compression.

Significantly for the purposes of this thesis, the acyclic chromatic number is a member of a large family of graph parameters, many of which are expressed as colouring numbers, that are bounded

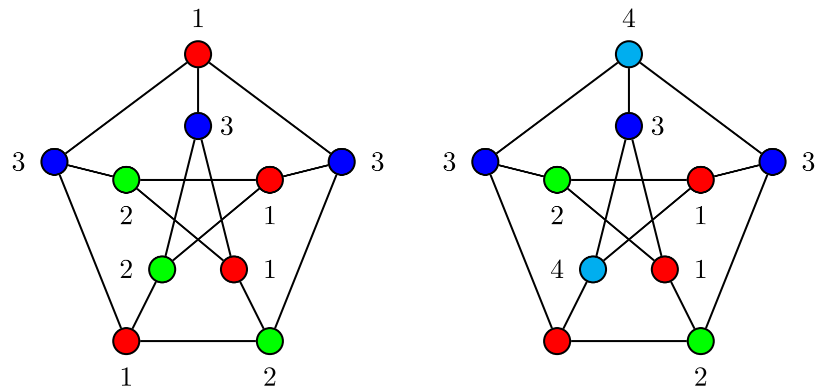

Figure 2.2: Two proper colourings of the Petersen Graph. The colouring on the left is not an acyclic colouring whereas the colouring on the right is an acyclic colouring.

above and below by functions of one another. Hence, one of these parameters, say acyclic chromatic number, is bounded for a family of graphs $\mathcal{G}$ if and only if every other parameter is also bounded on $\mathcal{G}$. As we will define in Section 2.5, the oriented chromatic number, which is the main object of this thesis, is also a parameter of this family.

Notably, the acyclic chromatic number of a graph G is not upper bounded by a function of the chromatic number of G. We note however, that the acyclic chromatic number of G' bounds the chromatic number of G, where G' is the 2-degenerate graph obtained by subdividing every edge in G exactly once. In particular Wood [84] proved the following result. For a graph G let the graph G' obtained by subdividing every edge of G exactly once be the 1-subdivision of G.

Theorem 2.4.1 ([84])**.** *Let G be a graph and G' be the* 1*-subdivision of G. Then*

$$\sqrt{\frac{\chi(G)}{2}} < \chi_a(G') \leq \max\{3, \chi(G)\}.$$

Notice that this result implies that there are bipartite graphs of arbitrarily large acyclic chromatic number. This is because as each graph G' is bipartite, if $\chi_a(G') \leq N$ for a constant N and all graphs G, then $\chi(G) \leq 2N(N-1)$ which is constant. Thus, as there exists graphs of arbitrarily large chromatic number, there exists bipartite graphs of arbitrarily large acyclic chromatic number. Although this connection between acyclic chromatic number and subdivision might at first seem curious but not profound, work by Dvořák [27] demonstrates that there is more to this relationship than meets the eye. Dvořák [27] does this by proving the following theorem.

Theorem 2.4.2 ([27])**.** *Let $\mathcal{G}$ be a class of graphs with bounded chromatic number. There exists a constant c such that $\chi_a(G) \leq c$ for all $G \in \mathcal{G}$ if and only if there exists a constant C such that for every subgraph H' of a graph $G \in \mathcal{G}$ where H' is the 1-subdivision of a graph H, with chromatic number $\chi(H) \leq C$.*

We note that subdivisions are natural to consider in minor closed families of graphs. Of course, as graphs embeddable on a fixed surface form a minor closed family, this result is of interest for this thesis. We note that as functions of the acyclic chromatic number are upper and lower bounds for the oriented chromatic number (more on this later), a similar theorem can be obtained for the oriented chromatic number of a class $\mathcal{G}$.

Let $\mathcal{F}$ be a family of connected simple graphs. Given an undirected graph $G = (V, E)$, a proper vertex colouring $\phi : V(G) \to \mathbb{N}$ is a $(2, \mathcal{F})$-*colouring* if and only if for all $i, j \in \mathbb{N}$, the 2-coloured graph $G[\phi^{-1}(i) \cup \phi^{-1}(j)]$ has no graph from $\mathcal{F}$ as a subgraph. We say ϕ is an $(2, \mathcal{F})$-k-*colouring* if the range of ϕ has cardinality k. The $(2, \mathcal{F})$-*chromatic number* of a simple graph G, denoted $\chi_{2,\mathcal{F}}(G)$, is the least integer k such that G admits a $(2, \mathcal{F})$-k-colouring. This is also sometimes called forbidden subgraph colouring.

Notice that taking $\mathcal{F}$ to be the set of all even cycles, a colouring ϕ is a $(2, \mathcal{F})$-colouring if and only if ϕ is an acyclic colouring. Similarly, other colouring parameters such as the star colouring can be expressed as special cases of $(2, \mathcal{F})$-colouring. See Figure 2.3 for an example of a $(2, \mathcal{F})$-colouring.

We note that many results for acyclic colouring can be generalised to $(2, \mathcal{F})$-colouring. For example Aravind and Subramanian [10] (which was originally released as a conference paper [8]) proved that for a family $\mathcal{F}$ of connected bipartite graphs on at least 3 vertices, such that the minimum number of edges in a graph in $\mathcal{F}$ is m, and at most s graphs in $\mathcal{F}$ have m edges, then

$$\chi_{2,\mathcal{F}}(G) \leq 64(m+1)^3 s \Delta^{1+\frac{1}{m-1}}$$

for all graphs G with maximum degree at most Δ. This generalises a result by Alon, Mcdiarmid, and Reed [3] for acyclic colouring that $\chi_a(G) = O(\Delta^{\frac{4}{3}})$ which we already mentioned, and a result by Fertin, Raspaud, and Reed [31] that $\chi_s(G) = O(\Delta^{\frac{3}{2}})$ where χ_s is the star chromatic number. Note that the same result was proven independently in [16].

As some final notes on forbidden subgraph colouring, we mention several results which are of interest, although they are not related to the results of the thesis. In a 2011 paper Aravind

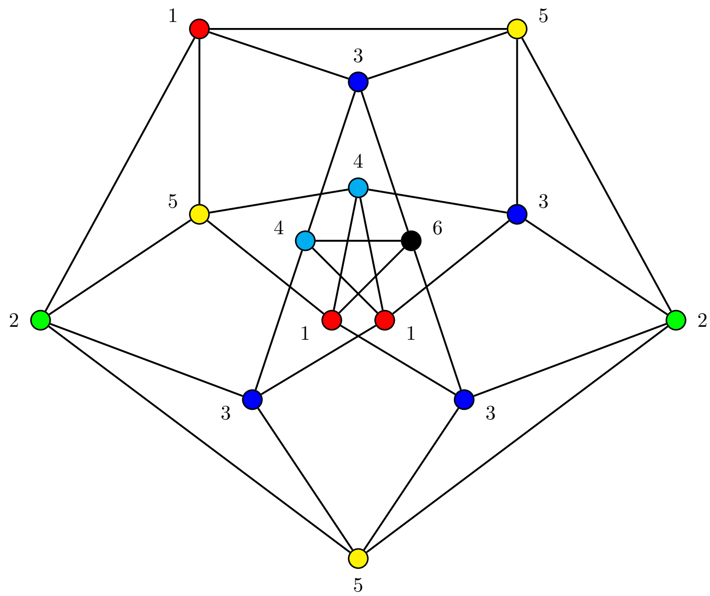

Figure 2.3: The line graph of the Petersen graph with a $(2, \mathcal{F})$-6-colouring, such that $\mathcal{F} = \{S_3, P_5\}$ where S_3 is the star with 3 leaves and P_5 is a 5-vertex path.

and Subramanian [9] explored other bounds related to $\chi_{2,\mathcal{F}}$, maximum degree, and the minimum number of edges in a graph in $\mathcal{F}$. In a different, but also interesting direction, Bradshaw [17] used forbidden subgraph colouring to prove that the game chromatic number of products of graphs is bounded as a function of the game chromatic number of its factors, answering a question of Zhu [86]. Finally, Kırtışoğlu and Özkahya [43] have recently studied $\chi_{2,\mathcal{F}}$ for graphs of bounded degree, focusing on the special case of $\mathcal{F} = \{P_k\}$, where P_k is the path on k-vertices.

2.5 Oriented Colouring

An *oriented colouring* of an oriented graph $G = (V, E)$ is a proper vertex colouring $c : V \to \mathbb{N}$ such that if $(u, v), (x, y) \in E$, then $c(u) = c(y)$ implies $c(v) \neq c(x)$. If the image of c has cardinality at most k, then we say G has an *oriented k-colouring*. For some examples consider Figure 2.4.

Equivalently, G has an *oriented k-colouring* if there exists a graph H of order at most k and a function $h : V(G) \to V(H)$ so that for all $(u, v) \in E(G)$, $(h(u), h(v)) \in E(H)$. Such a map h is called an *oriented homomorphism*. See Figure 2.5 for an example.

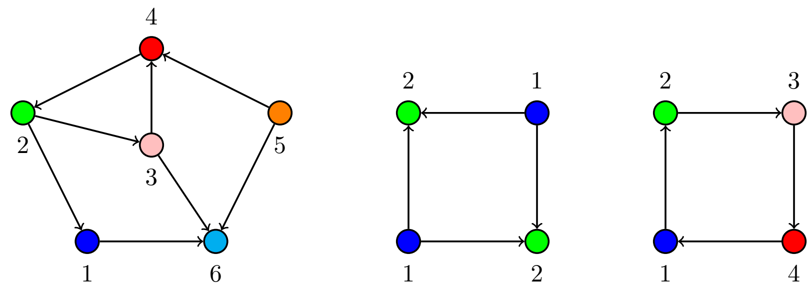

Figure 2.4: Three examples of oriented colourings (consider each component as its own graph).

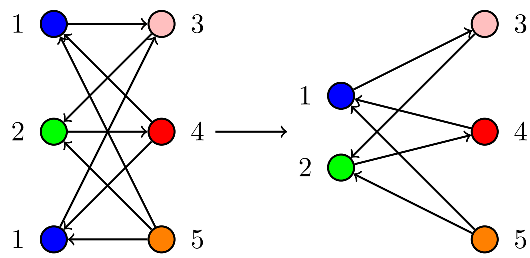

Figure 2.5: An example of an oriented homomorphism.

Meanwhile, the *oriented chromatic number* of an oriented graph G, denoted $\chi_o(G)$, or simply χ_o when the choice of G is obvious, is the least integer k so that G has an oriented k-colouring. We note that if G is a simple (non-oriented) graph, then $\chi_o(G)$ is the maximum oriented chromatic number of any orientation of G.

We recommend [77] as a survey of the oriented colouring literature. For completeness, and because [77] does not cover all of the background we require for this thesis, we include a short survey of the oriented colouring literature below.

This parameter was first introduced and studied by Courcelle [22] as means to encode a graph orientation as a vertex labelling. Since its inception, $\chi_o(G)$ has been extensively studied with the first major results coming from Raspaud and Sopena [68], who proved $\chi_o(G) \leq \chi_a(G)2^{\chi_a(G)-1}$ where $\chi_a(G)$ is the acyclic chromatic number of the underlying graph of G. As Borodin [12] had previously shown the acyclic chromatic number of a planar graph is at most 5, Raspaud and Sopena in fact proved that the oriented chromatic number of a planar graph is at most 80. A bound that has not been improved in the nearly 30 years since its publication, despite many efforts to do so

[77]. Moreover, it is unknown if there exists a planar graph that requires more than 18 colours in an oriented colouring [51, 53, 76, 77].

Significant efforts have been made to achieve partial results. A particular focus has been to give improved bounds on the oriented chromatic number of planar graphs with prescribed girth. Here the girth of a graph is the length of its shortest cycle. See Table 2.5 for a list of these bounds. A majority of the given upper bounds are proven using discharging arguments.

Girth $\geq$	Best Lower Bound	Best Upper Bound
3	18 [53]	80 [68]
4	11 [66]	40 [66]
5	7 [52]	16 [67]
6	7 [52]	11 [13]
7	6 [62]	7 [14]
8	5 [62]	7 [14]
11	5 [62]	6 [65]
12	5 [62]	5 [15]

Table 2.1: A list of current best upper and lower bounds on the oriented chromatic number of planar graphs with a given girth. Beside each bound we cite of the paper that proves it.

Elaborating on the relationship between oriented chromatic number and acyclic chromatic number, combining the effort of Raspaud and Sopena [68] and Kostochka, Sopena, and Zhu [46], it was shown that the oriented chromatic number is bounded above and below by functions of the acyclic chromatic number. This implies that the oriented chromatic number is also a function in the family of parameters bounded above and below by functions of the acyclic chromatic number mentioned in the previous section. The specifics of the bounds from [46, 68] are as follows.

Theorem 2.5.1 ([46, 68]). *If G is a graph and $\chi_a(G) = k$ and $\chi_o(G) = \ell$, then*

$$\ell \leq k2^{k-1}$$

and

$$k \leq \ell^{\lceil \log_2(\lceil \log_2 \ell \rceil + \frac{\ell}{2}) \rceil + 1}.$$

Corollary 2.5.2 ([46]). *If $\chi_o(G) = \ell \geq 4$, then*

$$\chi_a \leq \ell^2 + \ell^{3 + \lceil \log_2 \log_2 \ell \rceil}.$$

This upper bound for χ_o in χ_a is perhaps the most used bound on χ_o in the literature. Notably, there is some evidence that the upper bound for χ_o in Theorem 2.5.1 is close to tight.

Theorem 2.5.3 ([46]). *There are graphs with acyclic chromatic number k and oriented chromatic number greater than $2^{k-1} - 1$.*

Bounding the oriented chromatic number in terms of the maximum degree, $\Delta = \Delta(G)$, was first considered by Sopena [75] who showed $\chi_o(G) \leq (2\Delta - 1)4^{\Delta-1}$. This was later improved by Kostochka, Sopena, and Zhu [46] to $\chi_o(G) \leq 2\Delta^2 2^\Delta$, later being expanded on by Aravind and Subramanian [10] who showed $\chi_o(G) \leq 16\Delta d 2^d$, where d is the degeneracy of the graph G. Recall that degeneracy $d(G)$ or simply d when the choice of G is obvious, is the smallest integer k such that $\delta(H) \leq k$ for all subgraphs H of G. Notably, there is no way to eliminate the maximum degree term in the previous upper bound entirely. For example, given any simple graph G, the oriented subdivision H of G given by subdividing every edge and orienting the new edges to form 2-dipaths has $\chi_o(H) \geq \chi(G)$ despite H being 2 degenerate (see Figure 2.6). Interestingly, Wood [84] proved that if $\chi(G) \geq 9$, then $\chi_o(H) = \chi(G)$ where H is the graph obtained from G as before. This fact can be understood in the context of work by Dvořák [27] who demonstrated that all parameters bounded above and below by functions of the acyclic chromatic number (which the oriented chromatic number is [46, 68]) have their values closely related to this type of subdivision.

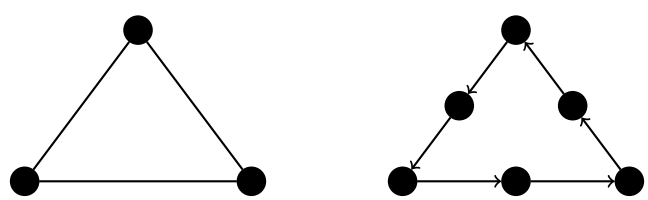

Figure 2.6: A triangle and a directed subdivision of a triangle where the new edges all form 2-dipaths.

More recently, the maximum degree bound was slightly improved for connected graphs to $\chi_o(G) \leq (\Delta - 1)^2 2^\Delta + 2$ by Das, Nandi, and Sen [23] in a more general context of connected (m, n)-colouring mixed graphs. In the same paper the authors also show that if G has degeneracy strictly less than its maximum degree, then the plus 2 term can be dropped implying $\chi_o(G) \leq (\Delta - 1)^2 2^\Delta$. Attempts to lower this bound for small values of Δ have also seen some notable progress and remain an active area of research [26, 28, 78].

An oriented clique is a graph $G = (V, E)$ such that $\chi_o(G) = |V|$. That is an oriented clique is a graph with no oriented homomorphism to a graph on fewer vertices than the graph itself. This naturally generalizes the notion of cliques, i.e. complete graphs, from proper colouring, as cliques are exactly those graphs with $\chi(G) = |V|$. Unlike cliques, oriented cliques of a given order are not unique. In fact, every graph with directed diameter 2 is an oriented clique. Furthermore it is not hard to verify that every oriented clique has directed diameter 2. For more on oriented cliques see [32, 45, 61, 73]

Given a graph G of Euler genus g, Kostochka et al. [46] observed that one may obtain an upper bound for $\chi_o(G)$ in terms of g by combining the inequality $\chi_o(G) \leq \chi_a(G)2^{\chi_a(G)-1}$ from Raspaud and Sopena [68] with a bound on the acyclic chromatic number of a graph with bounded Euler genus. The current best such bound is the estimate $\chi_a(G) = O(g^{4/7})$ of Alon, Mohar, and Sanders [3], which yields the upper bound $\chi_o(G) = 2^{O(g^{4/7})}$.

By generalizing Raspaud and Sopena's [68] acyclic chromatic number upper bound for the oriented chromatic number, Aravind and Subramanian [10] proved the following upper bound on oriented chromatic number in terms of $(2, \mathcal{F})$-chromatic number. Letting $\mathcal{F}$ be a family of connected graphs, let $\text{Forb}(\mathcal{F})$ be the set of all graphs H that contain no $F \in \mathcal{F}$ as a subgraph.

Theorem 2.5.4. *Let $\mathcal{F}$ be a family of connected graphs. Suppose there exists a integer t such that $\chi_o(H) \leq t$ for all $H \in Forb(\mathcal{F})$. Then, for any graph G with no subgraph in $Forb(\mathcal{F})$ and $\chi_{2,\mathcal{F}}(G) \leq k$,*

$$\chi_o(G) \leq kt^{k-1}.$$

Additionally, Aravind and Subramanian [10] proved that if $\mathcal{F}$ is a family of connected bipartite graphs on at least 4 vertices each with maximum degree 2, then $\chi_{2,\mathcal{F}}(G) = O(g^{\frac{m}{2m-1}})$, for all graphs G of Euler genus at most g, where m is the smallest number of edges in any member of $\mathcal{F}$. Using this result, and the previously stated upper bound on the oriented chromatic number, Aravind and Subramanian [10] showed that for any for every constant $\varepsilon > 0$, $\chi_o(G) = 2^{O\left(g^{\frac{1}{2}+\varepsilon}\right)}$. Aravind and Subramanian further conjectured that the ε in the exponent can be removed as follows:

Conjecture 2.5.5 ([10])**.** *If G is a graph of Euler genus g, then $\chi_o(G) = 2^{O(\sqrt{g})}$.*

We note that all previous upper bounds on the oriented chromatic number of a graph in terms of its Euler genus use the following proof strategy, originally appearing in [46]. First, a proper k-

colouring ψ of G is fixed, such that the two-coloured subgraphs of G under ψ satisfy some specific property, such as being acyclic or having components with few edges. Next, each vertex $v \in V(G)$ receives a new colour $\psi'(v)$ which is one of $f(k) \geq 2^{k-1}$ possible *shades* of the colour $\psi(v)$, where the precise value of $f(k)$ depends on the properties of ψ. Finally, it is argued that the new colouring ψ' is an oriented colouring of G, which implies that $\chi_o(G) \leq kf(k)$. We observe that if G has Euler genus g, then a proper colouring ψ of G may require $k = \Omega(\sqrt{g})$ colours. As the strategy outlined above requires at least 2^{k-1} shades to be allowed for each vertex, the bound in Conjecture 2.5.5 is best possible using this approach.

2.6 2-Dipath Colouring

Given an oriented graph $G = (V, A)$ a 2-*dipath colouring* $\phi : V \to \mathbb{N}$ is a proper colouring such that for each pair of vertices u, v if $1 \leq \text{dist}(u, v) \leq 2$, then $\phi(u) \neq \phi(v)$. Here distance is directed distance. We say ϕ is a 2-dipath ℓ-colouring if ϕ is a 2-dipath colouring and the range of ϕ has cardinality at most ℓ. Then the 2-dipath chromatic number of G, denoted $\chi_2(G)$, is the least integer t such that G admits a 2-dipath t-colouring. First proposed by Chen and Wang [56], a 2-dipath colouring is equivalent to a proper colouring of the directed square G^2 the of the graph G.

Notice that every oriented colouring is a 2-dipath colouring. This is because if ϕ is a colouring such that there exists a 2-dipath u, v, w with $\phi(u) = \phi(w)$, then ϕ is not an oriented colouring. From this it is clear that $\chi_o(G) \geq \chi_2(G)$. In particular, we can view $\chi_2(G)$ as a localized version of $\chi_o(G)$ as every oriented colouring is a 2-dipath colouring and 2-dipath colourings must only satisfy local constraints unlike oriented colourings. Perhaps the best example of this local versus global behaviour is that for all graphs G, $\chi_2(G) = \max\{\chi_2(C) : C \text{ is a connected component in } G\}$ whereas there exist graphs H such that $\chi_o(H) > \max\{\chi_o(C) : C \text{ is a connected component in } H\}$. For an example of such a graph H, see Figure 2.7.

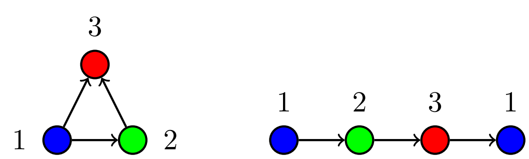

Figure 2.7: A digraph with two components is depicted, along with a 2-dipath colouring that uses 3-colours. Note that the oriented chromatic number of this graph is 4. Hence, not every 2-dipath colouring is an oriented colouring.

This relationship between oriented and 2-dipath colouring can be understood by a result of MacGillivray and Sherk [50], which characterises if a graph G admits a 2-dipath k-colouring based on the existence of an oriented homomorphism from G to a given target graph. As a result of this, MacGillivray and Sherk [50] achieve the following upper bound.

Theorem 2.6.1 ([50]). *If G has $\chi_2(G) = k$, then $\chi_o(G) \leq 2^{k-1}$.*

The proof of Theorem 2.6.1 proceeds by constructing an auxiliary graph H, such that G admits an oriented 2-dipath colouring if and only if G has an oriented homomorphism to H. As we will see in Section 3.2 and Section 4.2, this bound is far from tight for certain classes of sparse graphs. However, as we will explore in Section 3.1 and Section 3.3 the 2-dipath chromatic number and oriented chromatic number of a graph can be exponentially far apart. Given this, it is reasonable to expect that for general graphs Theorem 2.6.1 is close to tight.

The 2-dipath chromatic number of graphs with small maximum degree has also seen some attention in relation to the oriented chromatic number of the same graphs. This connection is best exemplified by Duffy in [25] who showed that all sub-cubic graphs are 2-dipath 7-colourable. This implies that the best upper bound for the oriented chromatic number of sub-cubic graphs, and the best upper bound for the 2-dipath chromatic number of sub-cubic graphs, differ by at most 1.

2.7 Injective Edge Colouring

Given a simple graph G, an *injective edge-colouring* of G is a function $\psi : E(G) \to \mathbb{N}$ such that if $\psi(e) = \psi(e')$ for distinct edges $e, e' \in E(G)$, then no third edge of G joins an endpoint of e to an endpoint of e'. In other words, if $\psi(e) = \psi(e')$, then e and e' are not at distance 1 and do not belong to a common triangle in G. The *injective chromatic index* of G is the minimum integer k for which G has an injective edge colouring $\phi : E(G) \to \{1, \ldots, k\}$. Note that an edge colouring $\phi : E(G) \to \{1, \ldots, k\}$ is injective if and only if each colour class of ϕ is an induced star forest in G. See Figure 2.8 for two examples of injective edge colourings.

The injective chromatic index of a graph was introduced by Cardoso, Cerdeira, Cruz, and Dominic [19] in 2015 as a theoretical model for a packet radio network problem, in which the goal is to assign communication frequencies to network node pairs in a way that eliminates secondary interference. These authors established bounds for the injective chromatic index of certain graph classes, including paths, cycles, and complete bipartite graphs. They also proved that computing a graph's injective chromatic index is NP-hard. The notion of injective chromatic index was reintroduced independently in 2019 by Axenovich, Dörr, Rollin, and Ueckerdt [11] under the name *induced star arboricity*, and these authors proved that the injective chromatic index can be bounded in terms of a graph's treewidth or acyclic chromatic number.

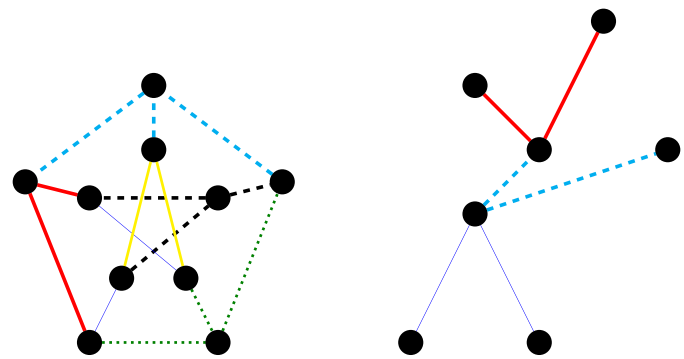

Figure 2.8: Two graphs with injective edge colourings.

Ferdjallah, Kerdjoudj, and Raspaud [30] first considered the problem of bounding a graph's injective chromatic index in terms of its maximum degree. They observed that by Brooks' Theorem, a graph G of maximum degree Δ satisfies $\chi'_{\text{inj}}(G) \leq 2(\Delta - 1)^2$. They also observed that injective edge colourings share a close relationship with *strong edge colourings*, which can be characterized as injective edge colourings in which any two incident edges receive distinct colours. The *strong chromatic index* of a graph G, written $\chi'_s(G)$, is the minimum number of colours required for a strong edge colouring of G, and hence for every graph G, $\chi'_{\text{inj}}(G) \leq \chi'_s(G)$. While a greedy argument shows that a graph G of maximum degree Δ satisfies $\chi'_s(G) \leq 2\Delta(\Delta-1)+1$, Erdős and Nešetřil [36] conjectured that every graph G of maximum degree Δ satisfies $\chi'_s(G) \leq \frac{5}{4}\Delta^2$, and this conjecture is still open. If the conjectured upper bound of Erdős and Nešetřil is correct, then it would be best possible, as the graph G obtained from C_5 by replacing each vertex with an independent set of size t and replacing each edge with a complete bipartite graph on the corresponding independent sets has maximum degree $\Delta = 2t$ and strong chromatic index exactly $\frac{5}{4}\Delta^2$. Currently, the best known upper bound for the strong chromatic index of a graph G of maximum degree Δ is $\chi'_s(G) \leq 1.772\Delta^2$, which was proven by Hurley, Kang, and de Verclos [38] using a more general argument that applies to graphs with sparse neighborhoods.

For d-degenerate graphs G with maximum degree Δ, Miao, Song, and Yu [55] used an edge ordering argument to show that upper bounds of the form $\chi'_{\text{inj}}(G) = O(\Delta^2)$ can be greatly improved to the following bound when d is small.

Theorem 2.7.1 ([55])**.** *If G is a d-degenerate graph of maximum degree Δ, then*

$$\chi'_{\text{inj}}(G) \leq (4d-3)\Delta - 2d^2 - d + 3.$$

One frequently studied class of degenerate graphs is the class of graphs with bounded Euler genus. If G is a graph of Euler genus g, then G is $O(\sqrt{g})$-degenerate, and hence Theorem 2.7.1 implies that $\chi'_{\text{inj}}(G) = O(\Delta\sqrt{g})$, where Δ is the maximum degree of G. In fact one can obtain an upper bound for $\chi'_{\text{inj}}(G)$ in terms of g that is independent of Δ. Indeed, Axenovich et al. [11] show that given a graph G,

$$\log_3(\chi_a(G)) \leq \chi'_{\text{inj}}(G) \leq 3\binom{\chi_a(G)}{2}, \tag{2.1}$$

where $\chi_a(G)$ is the acyclic chromatic number of G. Alon, Mohar, and Sanders [3] proved that if G is a graph of Euler genus g, then $\chi_a(G) = O(g^{4/7})$, implying that $\chi'_{\text{inj}}(G) = O(g^{8/7})$.

Using a deterministic approach, with some small assistance of the Lovász local lemma, the author, Bradshaw, and Xu [18] give a massive improvement on this bound by showing that for all graphs G with Euler genus at most g, $\chi'_{\text{inj}}(G) \leq (3 + o(1))g$. This bound is tight up to the choice of second order term when G is a clique. As a result, determining the best bounds for the injective chromatic index in terms of a graphs Euler genus is essentially resolved.

Using a probabilistic approach, Kostochka, Raspaud, and Xu [44] showed that given a graph G of maximum degree Δ, upper bounds of the form $\chi'_{\text{inj}}(G) = O(\Delta^2)$ can also be greatly improved when G has small chromatic number.

Theorem 2.7.2 ([44])**.** *If G is a graph of maximum degree Δ and chromatic number χ, then*

$$\chi'_{\text{inj}}(G) \leq (\chi - 1)\lceil 27\Delta \log \Delta \rceil.$$

The main idea of the proof of Theorem 2.7.2 is that given a graph G and an independent set X, a certain random procedure can find an induced star forest in G that contains many edges in the cut $[X, G \setminus X]$. By repeating this random procedure $O(\Delta \log \Delta)$ times, one can partition all edges in the cut $[X, G \setminus X]$ into $O(\Delta \log \Delta)$ induced star forests. By repeating this procedure for all but one colour class in a proper colouring of G, one obtains an injective edge colouring of G. Kostochka, Raspaud, and Xu [44] asked whether the $\log \Delta$ factor in Theorem 2.7.2 can be removed, and this question is still open, even in the case that G is bipartite.

Using a similar strategy, this bound was further improved by the author, Bradshaw, and Xu in [18] for graphs whose degeneracy is small relative to $\sqrt{\Delta}$. The improved bound is of the following form.

Theorem 2.7.3 ([18]). *If G is a d-degenerate graph of maximum degree $\Delta \geq 3$ and chromatic number χ, then*

$$\chi'_{\text{inj}}(G) \leq \lceil 4ed \log \Delta \rceil (2d+1)\chi.$$

Chapter 3

Graphs with Bounded Degree and Degeneracy

3.1 2-Dipath Colouring Graphs of Bounded Maximum Degree and Degeneracy

In this section we will upper bound the 2-dipath chromatic number of a graph in terms of its maximum degree and degeneracy. Our argument follows a greedy strategy. It is unclear if the following bound is tight, although it should be noted that bounds resulting from naive greedy strategies are rarely tight. We also note, that even if more complicated arguments are used to augment a greedy strategy, a greedy approach is inconsistent in producing tight upper bounds on colouring numbers.

For instance it is easy to prove, using Euler's formula, that every planar graph is 5-degenerate, implying that every planar graph is 6-colourable. However, we know that there is no planar graph with chromatic number greater than 4, so this greedy strategy is insufficient to prove a tight bound. Notably, for planar graphs this greedy strategy can be augmented by a structural argument to prove the 5-colour theorem, which states that every planar graph is 5-colourable. However, to prove the 4-colour theorem, that is the tight bound for the chromatic number of planar graphs, a much more complicated strategy is required.

Having said all of this, we note that the following upper bound is tight up to a constant factor. This is because there exists graph G with $\chi(G) = \Delta + 1$ where Δ is the maximum degree of G. Hence, the 1-subdivision of G, call it H, will have $\chi_o(H) = \chi_2(H) = \chi(G) = \Delta + 1$ whenever $\Delta \geq 8$ by a result from [84]. Note that this result from [84] was already discussed in Chapter 2.5. As Δ

will also be the maximum degree of H, while H is 2-degenerate, we conclude that our bound is at most 3 times the true value of a tight upper bound.

Theorem 3.1.1. *If $G = (V, E)$ is a graph with maximum degree Δ and degeneracy d, then*

$$\chi_2(G) \leq 2d\Delta - \Delta - d^2 + d + 1.$$

Proof. Let $G = (V, E)$ be a graph with maximum degree Δ and degeneracy d. By the definition of the degeneracy, there exists a vertex ordering $v_1, \dots, v_n$ of V, such that $|N(v_i) \cap \{v_1, \dots, v_{i-1}\}| \leq d$ for all $1 \leq i \leq n$. Fix i and suppose without loss of generality that $k = |N(v_i) \cap \{v_1, \dots, v_{i-1}\}|$. Notice that this implies that there are at most $k(\Delta - 1)$ vertices v_j, such that v_j is a neighbour of $v_t \in N(v_i) \cap \{v_1, \dots, v_{i-1}\}$. Furthermore, there are at most $(d-1)(\Delta - k)$ vertices v_j, $j < i$, such that v_j is a neighbour of $v_t \in N(v_i) \cap \{v_{i+1}, \dots, v_n\}$. Thus, v_i has at most $k(\Delta - 1) + (d-1)(\Delta - k) = (d+k)\Delta - \Delta - dk + k$ second neighbours in $\{v_1, \dots, v_{i-1}\}$.

Now we will colour G greedily as follows. Suppose $\phi : \{v_1, \dots, v_{i-1}\} \to \{1, \dots, 2d\Delta - \Delta - d^2 + d + 1\}$ is a colouring of $G[\{v_1, \dots, v_{i-1}\}]$ such that for all vertices v_j where $j \geq i$, if v_t and v_r are both neighbours of v_j and $r, t < i$, then $\phi(v_r) \neq \phi(v_t)$. We will prove that we can extend ϕ to colour v_i. As v_i was selected without loss of generality this will imply that G can be 2-dipath coloured using at most $2d\Delta - d + 1$ colours.

By assumption, for all $v_t, v_r \in N(v_i) \cap \{v_1, \dots, v_{i-1}\}$, $\phi(v_t) \neq \phi(v_r)$. Hence, adding v_i to $G[\{v_1, \dots, v_{i-1}\}]$ will not create any 2-dipaths between vertices of the same colour under ϕ. Hence, ϕ is a partial 2-dipath colouring of $G[\{v_1, \dots, v_i\}]$. Hence, if v_i receives a colour that is distinct from any of its already coloured neighbours, that is the vertices in $N(v_i) \cap \{v_1, \dots, v_{i-1}\}$, while also being distinct from all of the already coloured second neighbours of v_i, then this extended version of ϕ satisfies our assumptions.

Recall that v_i has most $k \leq d$ already coloured neighbours and at most $(d+k)\Delta - \Delta - dk$ already coloured second neighbours. Hence, if there are at least l colours available to colour v_i, where

$$l > (d+k)\Delta - \Delta - dk + k$$

then there exists a colour we can assign v_i to extend ϕ to colour $G[\{v_1, \dots, v_i\}]$ as required. By taking a first derivative in k and recalling that $\Delta \geq d$, we can see that $(d+k)\Delta - \Delta - dk + k$ is maximised when $k = d$. Hence, if we have at least l colours where

$$l > 2d\Delta - \Delta - d^2 + d$$

then there exists a colour we can assign v_i to extend ϕ. As we have assumed we have at least $2d\Delta - \Delta - d^2 + d + 1$ colours, this concludes the proof. □

From here we can get a nice corollary for graph families with bounded degeneracy. Recall that such families are of interest, given they include graph subdivisions, as well as graphs embeddable on a fixed surface (which have bounded degeneracy by Euler's formula).

Corollary 3.1.2. *If $\mathcal{G}$ is a graph family of graphs with bounded degeneracy, then for all $G \in \mathcal{G}$, $\chi_2(G) = O(\Delta)$.*

3.2 Bounds for Oriented Chromatic Number in 2-Dipath Chromatic Number

This section focuses on bounding the oriented chromatic number as a function of the 2-dipath chromatic number and degeneracy of a graph. The results in this section appear in the author's paper [21]. We recall from Chapter 2 that the oriented chromatic number cannot be bounded as a function of a graph's degeneracy, as there are 2-degenerate graphs with arbitrarily high oriented chromatic number. On the other hand, as was mentioned in Chapter 2, the oriented chromatic number of a graph G can be bounded in terms of the 2-dipath chromatic number of G. Specifically, for all graphs G, $\chi_2(G) \leq \chi_o(G) \leq 2^{\chi_2(G)}$ as proven in [50].

The focus of this section then is to show how for certain graphs with small degeneracy relative to their 2-dipath chromatic number, have oriented chromatic number much smaller than 2^{χ_2}. The main result of this section is the upper bound from Theorem 2.6.1 which proves that if G is a graph with degeneracy d satisfying that $d \ll \chi_2(G)$, then $\chi_o(G) \ll 2^{\chi_2(G)}$. This serves as the first significant improvement on the $\chi_o(G) \leq 2^{\chi_2(G)}$ upper bound for a large class of graphs. We will also see later in Chapter 4 that this bound can be modified to show that the oriented and 2-dipath chromatic number of large genus surfaces cannot be too far apart, which is non-trivial.

We borrow the following notation and definition from [21]. Given an oriented graph G, a vertex $v \in V(G)$, and an ordered vertex set $U = (u_1, \ldots, u_t) \subseteq N(v)$, we write $F(U, v, G)$ for the vector in $\{-1,1\}^t$ whose i^{th} entry is 1 if vu_i is an arc of G, and whose i^{th} entry is -1 if u_iv is an arc of G. Now, suppose H is an oriented k-partite graph with exactly N vertices in each partite set. Let the partite sets of H be called $P_1, \ldots, P_k$. We say that H is (k,t,N)-*full* if the following holds: for each value $i \in [k]$, each ordered subset $U = (u_1, \ldots, u_d) \subseteq \bigcup_{j \neq i} P_j$ of size t, and each vector $q \in \{-1,1\}^t$, there exists a vertex $x \in P_i$ such that $F(U,x,H) = q$.

To provide some intuition for the structure of a (k,t,N)-full graph, we construct a $(2,2,4)$-full graph. For a figure depicting this graph see Figure 3.1. As the structure of this graph may not be immediately clear, we also provide the adjacency matrix of the grpah pictured in Figure 3.1.

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Observe that this graph was constructed by considering the Hadamard matrix

$$H = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

and then generating the top right quarter of the matrix A from

$$\begin{pmatrix} H & -H \\ -H & H \end{pmatrix}.$$

Observe that this results in a $(2,2,4)$-full graph for the following reasons. First, and most obviously, the resulting graph is on 8 vertices. Second, we force A to be symmetric and only add edges to the top right quarter of A, implying that the resulting graph is bipartite. Third and finally, as H is a Hadamard matrix, for any columns i and j there exists rows r and s such that $x_{r,i} = x_{r,j}$ and

$x_{s,i} = -x_{s,j}$. The same is true if rows and columns are reversed. Hence, when we consider

$$\begin{pmatrix} H \\ -H \end{pmatrix}$$

for each pair of columns and each vector $q \in \{-1, 1\}^2$, there exits a row whose intersection with our fixed columns is q. By the symmetry of our construction this implies the same will be true for rows.

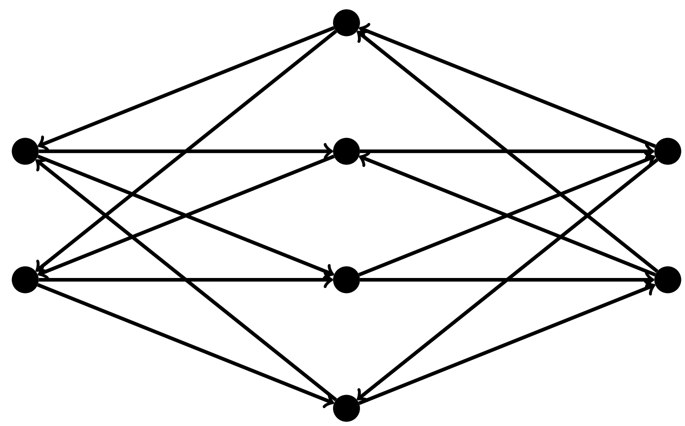

Figure 3.1: An example of a $(2, 2, 4)$-full graph.

It is unclear to the author if this strategy can be generalised to construct (k, d, N)-full graphs, for all k and d. If it is possible, then this would be interesting both for its own sake as a problem in design theory, and because (k, d, N)-full graphs are significant for oriented colouring. Without further ado, we explore exactly why these graphs are significant for oriented colouring.

Lemma 3.2.1. *Let K be a (k, d, N)-full graph. If G is a graph with $\chi_2(G) \leq k$ and degeneracy $d(G) \leq d$, then G has an oriented homomorphism to K.*

Proof. Let $G = (V, E)$ be a graph with $\chi_2(G) = k$ and degeneracy $d(G) = d$, and let K be a (k, d, N)-full graph with the partite sets $P_1, P_2, \ldots, P_k$. Let $\phi : V \to \{1, \ldots, k\}$ be a 2-dipath colouring of G. We will build an oriented homomorphism $h : G \to K$ satisfying $h(u) \in P_i$ if and only if $i = \phi(u)$.

Let $v_1, v_2, \ldots, v_n$ be a degeneracy ordering of G, that is $|N(v_j) \cap \{v_1, \ldots, v_{j-1}\}| \leq d$ for all $1 \leq j \leq n$. Suppose we have defined $h(v_j)$ for all $j < s$ satisfying our assumed condition that

$h(u) \in P_i$ if and only if $i = \phi(u)$. Let $A = N(v_s) \cap \{v_1, \dots, v_{s-1}\}$. By our choice of the degenerate ordering, $|A| \le d$. Now consider $h(A)$, the image of A under h. Notice that $|h(A)| \le |A| \le d$ and $h(u) \notin P_{\phi(v_s)}$ for all $u \in A$ given $\phi(u) \ne \phi(v_s)$ as ϕ is a proper colouring. Let $\vec{a} \in \{-1,1\}^{|A|}$ such that $F(A, v_s, G) = \vec{a}$. Then let $\vec{b} \in \{-1,1\}^{|h(A)|}$ be defined by $\vec{b}(h(u)) = \vec{a}(u)$. Note that as ϕ is a 2-dipath colouring of G, if $h(u) = h(w)$ for $u, w \in A$, then in G edges uv_s and wv_s are both oriented either from or towards v_s. So $\vec{b}$ is well defined.

Since K is (k, d, N)-full, there exists $x \in P_{\phi(v_s)}$ such that $F(h(A), x, K) = \vec{b}$. Let $h(v_s) = x$. It is clear that h is a homomorphism as required. □

Lemma 3.2.2. *If $k \ge 2$ and $t \ge \log_2 k$, then there exists a $(k, t, \frac{33}{10}t^2 2^t)$-full graph.*

Proof. Let $k \ge 2$ and $t \ge \log_2 k$. Notice that this implies $k \le 2^t$. Consider a random orientation of the complete k-partite graph $K = K_{N,\dots,N} = (P_1, \dots, P_k, E)$ where $N = \frac{33}{10}t^2 2^t$. Note each edge of K is oriented independently and uniformly. For each fixed value $i \in \{1, 2, \dots, k\}$ and a subset $A \subset \cup_{j \ne i} P_j$ satisfying $|A| = t$, let $X_{i,A}$ be the random variable

$$X_{i,A} := \sum_{\vec{a} \in \{-1,1\}^t} \mathbb{1}_{\forall v \in P_i, F(A,v,K) \ne \vec{a}}.$$

That is, $X_{i,A}$ counts the number of vectors $\vec{a} \in \{-1,1\}^k$ such that no vertex in P_i has orientation $\vec{a}$ with respect to A. Observe that the orientation of two edges between a vertex of A and two distinct vertices of P_i is independent. Furthermore, observe that this implies $X_{i,A} = 0$ is equivalent to a random function from a domain of size N to a codomain of size 2^t being surjective. Hence,

$$\mathbb{P}(X_{i,A} > 0) \le 2^t (1 - 2^{-t})^N \le 2^t e^{-2^{-t} N} = 2^t e^{-\frac{33}{10}t^2}$$

Applying the union bound,

$$\begin{aligned}
\mathbb{P}(\exists i, A, \text{ such that } X_{i,A} > 0) &\le k \binom{(k-1)N}{t} \mathbb{P}(X_{i,A} > 0) \\
&\le k \binom{(k-1)N}{t} 2^t e^{-\frac{33}{10}t^2} \\
&\le k \frac{(kN)^t}{t!} 2^t e^{-\frac{33}{10}t^2} \\
&\le 2k^{t+1} N^t e^{-\frac{33}{10}t^2}.
\end{aligned}$$

If $t \geq 1$, we can apply a logarithm to the above upper bound on the probability, and get

$$\begin{aligned} 1+(t+1)\log_2 k + t\log_2(N) - \frac{33}{10}\log_2(e)t^2 &\leq 1+(t+1)\log_2 2^t + t\log_2(\frac{33}{10}t^2 2^t) - \frac{33}{10}\log_2(e)t^2 \\ &\leq 1+(1+\log_2\frac{33}{10})t + 2\log_2(t)t + (1-\frac{33}{10}\log_2(e))t^2 \\ &< 0\,. \end{aligned}$$

We note that since $k \geq 2$, $t \geq \log_2 k \geq \log_2 2 = 1$. As this is a logarithm of an upper bound on $\mathbb{P}(\exists i, A, \text{ such that } X_{i,A} > 0)$, it follows for all $k \geq 2$ and $t \geq \log_2 k$, we have $\mathbb{P}(\exists i, A, \text{ such that } X_{i,A} > 0) < 1$. Thus, for all $k \geq 2$ $t \geq \log_2 k$ there exists a $(k, t, \frac{27}{10}t^2 2^t)$-full graph. ☐

We note that the coefficient of $\frac{33}{10}$ in Lemma 3.2.2 can be improved to $\frac{1}{\log_2(e)} + \varepsilon$, for any $\varepsilon > 0$, if we alter the statement to suppose that k (and therefore t) are sufficiently large. However, doing so would mean that Lemma 3.2.2 could not be applied to prove Theorem 3.2.3 unless a similar assumption about k and t being large is made. Thus, our result would not apply to all graphs. For example taking the coefficient to be 1 rather than 33/10 we must choose $t \geq 23$ rather than $t \geq 1$, which would force $k \geq 2^{23}$. We chose to set our coefficient at 33/10 as it the smallest “nice" fraction so that Theorem 3.2.3 applies to all graphs with at least 1 edge. See Table 3.2 for a longer list of possible improvements to the coefficient in Lemma 3.2.2 and Theorem 3.2.3, as well as an indication of the smallest k and t where these coefficients could be applied.

Theorem 3.2.3. *Let $k \geq 2$ be a fixed but arbitrary integer. Then for all integers $t \geq \log_2 k$ and for all graphs G with $d(G) \leq t$ and $\chi_2(G) \leq k$,*

$$\chi_o(G) \leq \frac{33}{10}kt^2 2^t.$$

Proof. Let $k \geq 2$ and $t \geq \log_2 k$ be fixed but arbitrary integers and let G be a graph with $d(G) \leq t$ and $\chi_2(G) \leq k$. By Lemma 3.2.2, there exists a $(k, t, \frac{33}{10}t^2 2^t)$-full graph K. Then Lemma 3.2.1 implies G has an oriented homomorphism to K. Thus, $\chi_o(G) \leq |V(K)| = \frac{33}{10}kt^2 2^t$ as desired. ☐

Significantly we can obtain a particularly nice corollary if the 2-dipath chromatic number of a class is not bounded, but the class has bounded degeneracy. This is because for graphs in such a class we can take $t = \max\{\log_2(\chi_2(G)), d\}$ where d is an upper bound on the degeneracy of every

Coefficient	$k \geq$	$t \geq$
33/10	2	1
3	4	2
5/2	4	2
2	8	3
3/2	2^6	6
1	2^{23}	23
3/4	2^{193}	193
7/10	2^{2310}	2310
139/200	2^{10135}	10135

Table 3.1: Improved coefficients for Theorem 3.2.3, the smallest k such that they apply, and the smallest t where this applies given the smallest k column.

graph in our class, which gives an asymptotic upper bound of $(\frac{33}{10} + o(1))k^2 \log_2^2(k)$ for the oriented chromatic number of graphs in the class. Recall that such classes exist as demonstrated in the introduction. Also recall such classes are notable given Theorem 2.4.2 from [27] and the fact that the oriented chromatic number is bounded above and below by a function of the acyclic chromatic number.

Corollary 3.2.4. *If $\mathcal{G}$ is a family of graphs with bounded degeneracy, then for $G \in \mathcal{G}$, $\chi_o(G) = O(\chi_2(G)^{2+o(1)})$ where the asymptotics are in $\chi_2(G)$.*

We note that Corollary 3.2.4 together with Theorem 3.1.1 imply that if $\mathcal{G}$ is a family of graphs with bounded degeneracy, then $\chi_o(G) = O(\Delta^{2+o(1)})$. As we will see in subsection 3.4, there exist graphs with oriented chromatic number at least exponential in maximum degree. Hence, Corollary 3.2.4 is notable, as it proves that any graphs with maximum degree Δ and large oriented chromatic number, must have large degeneracy. We note however that this upper bound is not best possible given that Corollary 3.5.4 implies that if $\mathcal{G}$ is a family of graphs with bounded degeneracy, then $\chi_o(G) = O(\Delta)$.

3.3 Oriented Colouring Graphs with Bounded Degree

In this section we will show that the oriented chromatic number can be bounded above as a function of a graph's maximum degree. The results in this section appear in the author's paper [21]. While the upper bound we give is exponential in maximum degree, which may seem far from being tight on first inspection, we note that in the next section we will show that there exists graphs whose oriented chromatic number is at least exponential in maximum degree.

We say a tournament T is (k,t)-comprehensive if for all $U \subset V(T)$ where $|U| = k$ and any $\vec{a} \in \{-1,1\}^k$ there exist at least t vertices $z \in V(T) \setminus U$ such that $F(U,z,T) = \vec{a}$. Such tournaments are notable as, like (k,t,N)-full graphs, they serve as convenient target graphs for an oriented homomorphism. Unlike (k,t,N)-full graphs, which were only recently defined in [21], (k,t)-comprehensive tournaments have been studied for some time, although not always by this name. These graphs have been particularly useful when oriented colouring graphs with bounded degree conditions such as planar graphs, graphs with bounded maximum degree, and graphs with bounded maximum average degree.

Also worthy of note, (k,t)-comprehensive tournaments are similar to another class of tournaments called tournaments with property S_k, first considered by Schütte and Erdős [29]. In fact, every $(k,1)$-comprehensive tournament has property S_k, although the converse may not be true. We note that Szekeres and Szekeres in [81] proved that every tournament with property S_k is of order at least $(k+2)2^{k-1}-1$. Thus, every $(k,1)$-comprehensive graph is of order at least $(k+2)2^{k-1}-1$. As we will see, this is close to being tight. For an example of a $(2,1)$-comprehensive tournament see Figure 3.3. Notice that most of the known examples of $(k,1)$-comprehensive tournaments are Payley tournaments. This is no coincidence as Graham and Spencer [34] proved that for every k there exists a sufficiently large Payley tournament with property S_k. The computational evidence suggests that all Payley tournaments with property S_k are also $(k,1)$-comprehensive, although this has not been proven.

Lemma 3.3.1. *If T is a (k,t)-comprehensive tournament where $k \geq 2$, then T is a $(k-1,2t)$-comprehensive graph.*

Proof. Suppose that T is a (k,t)-comprehensive tournament where $k \geq 2$. Let $A = \{v_1,\ldots,v_{k-1}\} \subset V(T)$ be a fixed but arbitrary and let $\vec{a} = (x_1,\ldots,x_{k-1}) \in \{-1,1\}^{k-1}$. Next, let $u \in V(T) \setminus A$ and $B = \{v_1,\ldots,v_{k-1},u\}$. As T is (k,t)-comprehensive there are at least t vertices z such that $F(B,z,T) = \vec{a}_+$, and at least t vertices w such that $F(B,w,T) = \vec{a}_-$, where $\vec{a}_+ = (x_1,\ldots,x_{k-1},1)$ and $\vec{a}_- = (x_1,\ldots,x_{k-1},-1)$. For any such z or w, $F(A,z,T) = F(A,w,T) = \vec{a}$, so there are at least $2t$ vertices z such that $F(A,z,T) = \vec{a}$. As our choice of A and $\vec{a}$ was arbitrary we conclude that T is $(k-1,2t)$-comprehensive as required. □

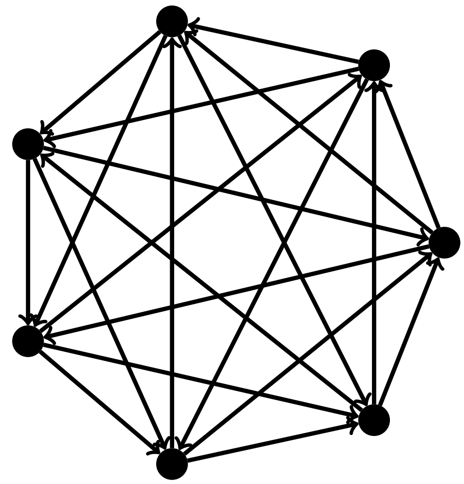

Figure 3.2: The Cayley graph $Cayley(\mathbb{Z}/7\mathbb{Z}; \{1,2,4\})$ which is a smallest $(2,1)$-comprehensive graph.

Lemma 3.3.2. *Let G be a graph of maximum degree $\Delta \geq 2$ with degeneracy $d \leq \Delta - 1$. If T is a $(\Delta-1,\Delta)$-comprehensive tournament, then G has an oriented homomorphism to T.*

Proof. Let $G = (V,E)$ and T be as in the statement of the lemma. Let $v_1, v_2, \ldots, v_n$ be a fixed degeneracy ordering of V. We define $h : V \to V(T)$ inductively on the degeneracy ordering of V such that it will satisfy

(1) $h|_{\{v_1,\ldots,v_i\}}$ is a homomorphism from $G[\{v_1,\ldots,v_i\}]$ to T, and

(2) for all v_j where $j > i$, $h(v_r) \neq h(v_s)$ for all $v_r, v_s \in N(v_j) \cap \{v_1,\ldots,v_i\}$.

We can define $h(v_1)$ arbitrarily as it will satisfy (1) and (2) trivially. Suppose $h|_{\{v_1,\ldots,v_i\}}$ is already defined and let $A = N(v_{i+1}) \cap \{v_1,\ldots,v_i\}$. By definition of degeneracy, $|A| \leq \Delta - 1$ and by (2), for all distinct $v_r, v_s \in A$, $h(v_r) \neq h(v_s)$. This implies that we need not be concerned about $h(v_r) = h(v_s)$ where v_r and v_s have different orientations to v_{i+1}. By our choice of T and Lemma 3.3.1 there are at least $2^{\Delta-1-|A|}\Delta$ vertices $z \in V(T)$ such that $F(h(A), z, T) = F(A, v_{i+1}, T)$. As T is of order at least $1 - \Delta + \Delta^2$, and as for any value of $|A| \geq 1$, there are at most $(\Delta - |A|)(\Delta - 1) < 2^{\Delta-1-|A|}\Delta$ vertices v_j, where $j \leq i$ and v_j has a common neighbour with v_{i+1} in $\{v_{i+2},\ldots,v_n\}$, given T is $(1, 2^{\Delta-2}\Delta)$-comprehensive by Lemma 3.3.1. It follows that there is a vertex $z \in V(T)$ such that for all these v_j, $h(v_j) \neq z$ and $F(h(A), z, T) = F(A, v_{i+1}, T)$. Choose such a z in T and let $h(v_{i+1}) = z$. Clearly,

$h|_{\{v_1,\ldots,v_i,v_{i+1}\}}$ satisfies (1) and (2). As the resulting mapping is a homomorphism, the lemma is proved. □

We will use a probabilistic argument to show that there exists certain (k,t)-comprehensive graphs. In order to do so we require the following Chernoff bound.

Lemma 3.3.3. *[[59] Theorem 4.2] Let X be a random variable with binomial distribution $Bin(n,p)$ where $0 < p < 1$ and $0 < \delta \leq 1$. Then,*

$$\mathbb{P}(X < (1-\delta)\mu) \leq \exp(-\mu\delta^2/2)$$

where $\mu := \mathbb{E}(X)$.

Lemma 3.3.4. *Let $\varepsilon > 0$ be a fixed constant. There exists an integer N, depending only on ε, such that for all $k \geq N$, there exists a $(k-1,k)$-comprehensive tournament of order $\lceil(\ln 2+\varepsilon)k^2 2^k\rceil$.*

Proof. Let $T = (V, E)$ be a random orientation of the complete graph on n vertices such that each edge is assigned an orientation uniformly and independently. We will choose the value of n later. Let $\vec{a} \in \{-1,1\}^{k-1}$ and $U \subset V$ such that $|U| = k-1$ be fixed but arbitrary. Let $X_{U,\vec{a}}$ be the random variable which counts the number of vertices $z \in V \setminus U$ such that $F(U, z, T) = \vec{a}$. It follows that the expectation $\mu := \mathbb{E}(X_{U,\vec{a}}) = (n-k+1)2^{1-k}$.

Letting $\delta = 1 - \frac{k}{\mu}$, observe that Lemma 3.3.3 implies

$$\begin{aligned} \mathbb{P}(X_{U,\vec{a}} < (1-\delta)\mu) &= \mathbb{P}(X_{U,\vec{a}} < k) \\ &\leq \exp(-\mu\delta^2/2) \\ &= \exp(k - \frac{\mu}{2} - \frac{k^2}{2\mu}) \end{aligned}$$

whenever $0 < \delta \leq 1$.

Now let $n = (\ln 2+\varepsilon)k^2 2^k$. Then,

$$\begin{aligned} \mu &= ((\ln 2+\varepsilon)k^2 2^k - k + 1)2^{1-k} \\ &= 2(\ln 2+\varepsilon)k^2 + \frac{1-k}{2^{k-1}} \end{aligned}$$

implying that for $k \geq 2$, $0 < \delta = 1 - \frac{k}{\mu} \leq 1$ as required to apply Lemma 3.3.3. Next observe that this implies for $k \geq 2$,

$$2(\ln 2 + \varepsilon)k^2 - 1 \leq \mu \leq 2(\ln 2 + \varepsilon)k^2 + 1.$$

Hence, we can conclude that for $k \geq 2$,

$$\begin{aligned} \mathbb{P}(X_{U,\vec{a}} < k) &\leq \exp(1/2 + k - (\ln 2 + \varepsilon)k^2 - \frac{k^2}{4(\ln 2 + \varepsilon)k^2 + 2}) \\ &= \exp((o(1) - \ln 2 - \varepsilon)k^2) \end{aligned}$$

where the $o(1)$ is a function of k. Applying the union bound over all choices of U and $\vec{a} \in \{-1, 1\}^{k-1}$, we arrive at the following inequalities valid for $k \geq 5$,

$$\begin{aligned} \mathbb{P}(\exists U, \vec{a},\ \textit{such that } X_{U,\vec{a}} < k+1) &\leq \binom{n}{k-1} 2^{k-1} \mathbb{P}(X_{U,\vec{a}} < k+1) \\ &\leq \binom{n}{k-1} 2^{k-1} \exp(-(\ln 2 + \varepsilon - o(1))k^2) \\ &\leq n^k \exp((o(1) - \ln 2 - \varepsilon)k^2) \\ &\leq ((\ln 2 + \varepsilon)k^2 2^k + 1)^k \exp((o(1) - \ln 2 - \varepsilon)k^2) \\ &= \exp((\ln 2 + o(1))k^2 + (o(1) - \ln 2 - \varepsilon)k^2) \\ &= \exp((o(1) - \varepsilon)k^2) \to 0 \end{aligned}$$

as $k \to \infty$, given the $o(1)$ is again in terms of k. Furthermore, note that all the functions captured by the $o(1)$ are monotone when $k \geq 5$, so we conclude that there exists an integer N depending on ε such that for all $k \geq N$, $\mathbb{P}(\exists U, \vec{a},\ \textit{such that } X_{U,\vec{a}} < k) < 1$.

Therefore, there exists an N such that for all $k \geq N$, there is a positive probability that T is $(k-1, k)$-comprehensive. Hence, for all $k \geq N$ there must exist a $(k-1, k)$-comprehensive tournament as desired. □

Theorem 3.3.5. *Let $\varepsilon > 0$ be a fixed constant. There exists an integer N depending only on ε such that for all $k \geq N$*

1. *if G is a graph with $d(G) < k$ and $\Delta(G) \leq k$, then $\chi_o(G) \leq \lceil (\ln 2 + \varepsilon)k^2 2^k \rceil$, and*
2. *if G is a connected graph with $\Delta(G) \leq k$, then $\chi_o(G) \leq \lceil (\ln 2 + \varepsilon)k^2 2^k \rceil + 2$, and*

3. *if G is a graph with $\Delta(G) \leq k$, $\chi_o(G) \leq 2\lceil(\ln 2+\varepsilon)(k+1)^2 2^k\rceil$.*

Proof. Let $\varepsilon > 0$ be a fixed constant and let N be the least integer that guarantees for all $k \geq N$, the existence of a $(k-1, k+1)$-comprehensive tournament of order $\lceil(\ln 2+\varepsilon)k^2 2^k\rceil$. Note that such an N exists by Lemma 3.3.4.

Case.1: G is a graph with $d(G) < k$ and $\Delta(G) \leq k$. Then Lemma 3.3.2 implies that G has an oriented homomorphism to any $(k-1, k+1)$-comprehensive tournament. By the choice of $k \geq N$, there exists a $(k-1, k+1)$-comprehensive tournament of order $\lceil(\ln 2+\varepsilon)k^2 2^k\rceil$. Hence, $\chi_o(G) \leq \lceil(\ln 2+\varepsilon)k^2 2^k\rceil$ as desired.

Case.2: G is a connected graph with $\Delta(G) \leq k$. Then either $d(G) < k-1$, or $d(G) = \Delta(G) = k$. In the former case the result follows by Case.1. Suppose then that G is $d(G) = \Delta(G) = k$. It is well known that a connected graph has $d = k$ if and only if it is k-regular. Thus, G is k-regular. Let $e = (u, v) \in E$ be fixed but arbitrary and let $H = G - e$. Then, $d(H) < \Delta(H) = \Delta(G) = k$. By the argument in Case 1, there is an oriented colouring $\phi_0 : V(H) \to \{1, \ldots, \lceil(\ln 2+\varepsilon)k^2 2^k\rceil\}$ of H. Let ϕ be a vertex colouring of G such that for all $w \neq u, v$, $\phi(w) = \phi_0(w)$, $\phi(u) = \lceil(\ln 2+\varepsilon)k^2 2^k\rceil + 1$, and $\phi(v) = \lceil(\ln 2+\varepsilon)k^2 2^k\rceil + 2$. As ϕ_0 is an oriented colouring it is easy to verify that ϕ is also an oriented $\lceil(\ln 2+\varepsilon)k^2 2^k\rceil + 2$-colouring. Hence, $\chi_o(G) \leq \lceil(\ln 2+\varepsilon)k^2 2^k\rceil + 2$ as desired.

Case.3: G is a graph with $\Delta(G) \leq k$. If $d(G) < k$, then we are in Case.1, thus, $d(G) = \Delta(G) = k$. Now, by Lemma 3.3.4 there exists a $(k, k+2)$-comprehensive tournament of order $2\lceil(\ln 2+\varepsilon)(k+1)^2 2^k\rceil$. Then by Lemma 3.2.1 G has an oriented homomorphism to this tournament, which implies $\chi_o(G) \leq 2\lceil(\ln 2+\varepsilon)(k+1)^2 2^k\rceil$ as desired. □

We note that by examining the bounds in Lemma 3.3.4 it can be observed that letting $k \geq 22$ is sufficient to let $\varepsilon = 3/10$ which improves the prior best known coefficients, see [23, 46], in every case of Theorem 3.3.5. However this improvement is somewhat marginal, so it is natural to ask how large k must be to reach a more significant improvement. Unfortunately the functions from Lemma 3.3.4 grow so fast that verifying this is non-trivial. See Table 3.3 for a short list of smallest values k where we may apply a given natural choices of ε, which we verified using a computer.

ε	$k \geq$
1	4
1/2	11
2/5	15
3/10	22
11/40	25
1/4	28

Table 3.2: Values of ε that appear in the coefficients for Theorem 3.3.5 in the first column with the smallest k such that these coefficients can be used appearing in the second column.

3.4 Lower Bounding the Oriented Chromatic Number in Maximum Degree

In this section we will present the current best lower bound for the oriented chromatic number in terms of maximum degree. We note that these bounds are not the work of the author. These results are included to provide context for the original contributions of the author in other sections. All of the results in this section are the work of Kostochka, Sopena, and Zhu [46].

Including these results in this thesis rather than referring to them with reference to [46] is done for two reasons. First, the ideas used in this lower bound are profound and interesting, while also not appearing in any of the novel contributions by the author, which are included in this thesis. Second, the original proofs for these results, while certainly correct, are extremely short to the point of being unclear on first inspection. The author has made an effort to expand these proofs, so that the very nice ideas they use can be more easily understood.

Lemma 3.4.1 ([46]). *If $G = (V, E)$ is a simple graph with n vertices and m edges and $\chi_o(G) \leq k$, then*

$$2^{\binom{k}{2}} k^n \geq 2^m.$$

Proof. Let $G = (V, E)$ be a labelled simple graph with n vertices and m edges. There are 2^m different orientations of G. Let S be the set of all orientations of G. Let $\phi : V \to \{1, \ldots, k\}$ be a fixed but arbitrary k-colouring of G. Notice here that we do not assume ϕ is an oriented colouring, or even a proper k-colouring of G.

Let $G' \in S$ be a fixed orientation of G. Observe that if ϕ is an oriented colouring of G', then this implies there is a tournament T' of order k and an oriented homomorphism from G' to T', call it h, such that labelling the vertices of T' by $1, \ldots, k$; for all $u \in V$, $h(u) = \phi(u)$. Now let G'' be an orientation of G distinct from G' where ϕ is also an oriented colouring of G''. Again this implies there exists a tournament, call it T'' whose vertices are labeled $1, \ldots, k$ such that the map h defined by for all $u \in V$, $h(u) = \phi(u)$, is an oriented homomorphism, this time from G'' to T''.

We claim that T' and T'' are distinct labeled tournaments. For the sake of contradiction suppose not. Observe that as G' and G'' are distinct orientations of the labeled graph G, there exists an edge joining vertices u and v in G, where $(u, v) \in E(G')$ and $(v, u) \in E(G'')$. As the map h is an oriented homomorphism for G' and G'' to T' and T'', respectively, this implies that $(h(u), h(v)) \in E(G')$ and $(h(v), h(u)) \in E(G'')$. But this contradicts our assumption that T' and T'' are the same labeled tournament. Hence, T' and T'' are distinct labeled tournaments as desired.

As G' and G'' are distinct orientations chosen without loss of generality, this implies that each orientation G' of G which satisfies that ϕ is an oriented colouring of G' corresponds to a unique k vertex labelled tournament. Hence, as ϕ was chosen without loss of generality, for all k-colourings of G, ϕ is an oriented colouring for at most $2^{\binom{k}{2}}$ orientations of G, given there are $2^{\binom{k}{2}}$ distinct labelled tournaments of order k.

Recall that $\chi_o(G) \leq k$ implies that every orientation of G admits an oriented k-colouring and there are 2^m orientations of G. As there are at most k^n, k-colourings of G, we conclude that if $\chi_o(G) \leq k$, then

$$2^{\binom{k}{2}} k^n \geq 2^m .$$

This concludes the proof. □

We note that the inequality given in Lemma 3.4.1 may seem somewhat awkward, however it can be manipulated to a great effect. Having now established some means to bound the oriented chromatic number as a function of n - the order of a simple graph G, and m - the size of a simple graph G, we are prepared to give our lower bound for the oriented chromatic number in terms of maximum degree.

Theorem 3.4.2 ([46])**.** *Let $\Delta \geq 2$ be an integer. There exists a graph $H = (V, E)$ with maximum degree Δ such that*

$$\chi_o(H) \geq 2^{\Delta/2}.$$

Proof. Let H be a Δ-regular simple graph on n vertices and let $k = \chi_o(H)$. Suppose without loss of generality that H is chosen to maximise k over all Δ-regular simple graph on n vertices. Then

H has $\frac{\Delta n}{2}$ edges by the handshaking lemma. Hence, Lemma 3.4.1 implies that

$$2^{\binom{k}{2}} k^n \geq 2^{\Delta n/2}.$$

Taking the n^{th} root of both sides, gives

$$2^{\binom{k}{2}/n} k \geq 2^{\Delta/2}.$$

Recall from Theorem 3.3.5 that as the oriented chromatic number of H is bounded above as a function of Δ, independent of n, we can choose n to be arbitrarily large relative to k, given Δ is constant. Suppose n is sufficiently large relative to k, say $n \geq 100^k$. Then,

$$\begin{aligned} 2k &> 2^{\binom{k}{2}100^{-k}} k \\ &= 2^{\binom{k}{2}/n} k \\ &\geq 2^{\Delta/2}. \end{aligned}$$

Hence, $k > 2^{\frac{\Delta}{2}-1}$. Given k is bounded, by taking the limit of n to infinity the same argument implies that for all $\varepsilon > 0$, $(1+\varepsilon)k > 2^{\Delta/2}$ as k is constant. As k is integer valued this implies $k \geq 2^{\Delta/2}$. So for a sufficiently large choice of n, there exists a graph with maximum degree Δ, and oriented chromatic number $k \geq 2^{\Delta/2}$. This completes the proof. □

3.5 Oriented Colouring Graphs with Bounded Degeneracy

In this section we will examine how much the bound given in Theorem 3.3.5 can be improved when a graph has degeneracy d, much smaller than its maximum degree Δ. Of course the phrase much smaller here is somewhat vague. We note that this is deliberate, as we intend to prove bounds which hold for all values of d relative to Δ. As a result of this generality, the statement of our bounds becomes somewhat more complicated compared to earlier sections. To combat this we make efforts to summarise the implications of our results at the end of this section. The lemmas and theorems in this section appear in the author's paper [21].

Lemma 3.5.1. *Let G be a graph of maximum degree Δ and degeneracy d. If T is a $(d, d\Delta)$-comprehensive tournament, then G has an oriented homomorphism to T.*

Proof. Let $G = (V, E)$ and T be as in the statement of the lemma. Let $v_1, v_2, \ldots, v_n$ be a fixed degeneracy ordering of V. We define $h : V \to V(T)$ inductively on the degeneracy ordering of V such that it will satisfy

(1) $h|_{\{v_1,\ldots,v_i\}}$ is a homomorphism from $G[\{v_1, \ldots, v_i\}]$ to T, and

(2) for all v_j where $j > i$, $h(v_r) \neq h(v_s)$ for all $v_r, v_s \in N(v_j) \cap \{v_1, \ldots, v_i\}$.

We can define $h(v_1)$ arbitrarily as it will satisfy (1) and (2) trivially. Suppose $h|_{\{v_1,\ldots,v_i\}}$ is already defined and let $A = N(v_{i+1}) \cap \{v_1, \ldots, v_i\}$. By definition of degeneracy $|A| \leq d$ and by (2), for all distinct $v_r, v_s \in A$, $h(v_r) \neq h(v_s)$. This implies that we need not be concerned about $h(v_r) = h(v_s)$ where v_r and v_s have different orientations to v_{i+1}. By our choice of T there are at least $d\Delta$ vertices $z \in V(T)$ such that $F(h(A), z, T) = F(A, v_{i+1}, T)$. As there are at most $(\Delta - |A|)(d - 1) < d\Delta$ vertices v_j where $j \leq i$ and v_j has a common neighbour with v_{i+1} in $\{v_{i+2}, \ldots, v_n\}$, and as T is of order at least $1 - \Delta + d\Delta$ given T is $(1, 2^{d-1}d\Delta)$-comprehensive by Lemma 3.3.1. It follows that there is a vertex $z \in V(T)$ such that for all these v_j, $h(v_j) \neq z$ and $F(h(A), z, T) = F(A, v_{i+1}, T)$. Choose such a z in T and let $h(v_{i+1}) = z$. Clearly, $h|_{\{v_1,\ldots,v_i,v_{i+1}\}}$ satisfies (1) and (2). As the resulting mapping is a homomorphism, the lemma is proved. □

Lemma 3.5.2. *Let $\alpha : \mathbb{N} \to (0, 1]$ be a monotone increasing function such that $\alpha(k)k^2 \to \infty$ as $k \to \infty$, and $\alpha(k)k \in \mathbb{N}$ for all k. There exists an integer N, depending on α, such that for all $k \geq N$, there is a $(\alpha(k)k, \alpha(k)k^2)$-comprehensive tournament of order $\lceil (2\alpha(k) \ln 2 + 2)\alpha(k)k^2 2^{\alpha(k)k} \rceil$.*

Proof. Let $\alpha : \mathbb{N} \to (0, 1]$ be a bounded monotone function such that $\alpha(k)k^2 \to \infty$ as $k \to \infty$, and $\alpha(k)k \in \mathbb{N}$ for all k. As in Lemma 3.3.4, we suppose T is a random tournament. Let $U \subseteq V$ with $|U| = \alpha(k)k$ be fixed but arbitrary. Let $X_{U,\vec{a}}$ be the random variable which counts the number of vertices $z \in V \setminus U$ such that $F(U, z, T) = \vec{a}$, where $\vec{a} \in \{-1, 1\}^{\alpha(k)k}$. Then $\mu := \mathbb{E}(X_{U,\vec{a}}) = (n - \alpha(k)k)2^{-\alpha(k)k}$.

Applying Lemma 3.3.3 with $\delta = 1 - \frac{\alpha(k)k^2}{\mu}$, we see that

$$\begin{aligned}
\mathbb{P}(X_{U,\vec{a}} < (1 - \delta)\mu) &= \mathbb{P}(X_{U,\vec{a}} < \alpha(k)k^2) \\
&\leq \exp(-\mu\delta^2/2) \\
&= \exp(\alpha(k)k^2 - \frac{\alpha^2(k)k^4}{2\mu} - \frac{\mu}{2})
\end{aligned}$$

whenever $0 < \delta \leq 1$. Let $n = (2\alpha(k)\ln 2 + 2)\alpha(k)k^2 2^{\alpha(k)k}$, then

$$\mu = (2\alpha(k)\ln 2 + 2)\alpha(k)k^2 - \frac{\alpha(k)k}{2^{\alpha(k)k}}$$

which, given our assumption that α is monotone and $\alpha(k)k^2 \to \infty$ as $k \to \infty$, implies that for large enough k depending on α, we have $0 < \delta \leq 1$ as required by Lemma 3.3.3. Also observe that for large enough k, depending on α,

$$(2\alpha(k)\ln 2 + 2)\alpha(k)k^2 - 1 \leq \mu \leq (2\alpha(k)\ln 2 + 2)\alpha(k)k^2 + 1.$$

Hence, for large enough k, depending on α,

$$\begin{aligned}
\mathbb{P}(X_{U,\vec{a}} < \alpha(k)k^2 + 1) &\leq \exp(\alpha(k)k^2 - \frac{\alpha^2(k)k^4}{2\mu} - \frac{\mu}{2}) \\
&\leq \exp(\frac{1}{2} + \alpha(k)k^2 - \frac{\alpha^2(k)k^4}{2(2\alpha(k)\ln 2 + 2)\alpha(k)k^2 + 2} - \frac{(2\alpha(k)\ln 2 + 2)\alpha(k)k^2}{2}) \\
&= \exp((\frac{2 - (2\alpha(k)\ln 2 + 2)}{2} - \frac{1}{4\alpha(k)\ln 2 + 4} + o(1))\alpha(k)k^2) \\
&\leq \exp((o(1) - \alpha(k)\ln 2 - \frac{1}{4\ln 2 + 4})\alpha(k)k^2)
\end{aligned}$$

where the asymptotics are in k. Now applying the union bound and choosing k to be sufficiently large with respect to α,

$$\begin{aligned}
&\mathbb{P}(\exists U, \vec{a}, \text{ such that } X_{U,\vec{a}} < \alpha(k)k^2 + 1) \\
\leq\ & \binom{n}{\alpha(k)k} 2^{\alpha(k)k}\mathbb{P}(X_{U,\vec{a}} < \alpha(k)k^2) \\
\leq\ & \frac{n^{\alpha(k)k}}{\alpha(k)k!} 2^{\alpha(k)k} \exp((o(1) - \ln 2\alpha(k) - \frac{1}{4\ln 2 + 4})\alpha(k)k^2) \\
\leq\ & ((2\alpha(k)\ln 2 + 2)\alpha(k)k^2 2^{\alpha(k)k})^{\alpha(k)k} \exp((o(1) - \ln 2\alpha(k) - \frac{1}{4\ln 2 + 4})\alpha(k)k^2) \\
\leq\ & 2^{(1+o(1))\alpha^2(k)k^2} \exp((o(1) - \ln 2\alpha(k) - \frac{1}{4\ln 2 + 4})\alpha(k)k^2) \\
=\ & \exp((o(1) - \ln 2\alpha(k) - \frac{1}{4\ln 2 + 4})\alpha(k)k^2 + (\alpha(k)\ln 2 + o(1))\alpha(k)k^2) \\
=\ & \exp((o(1) - \frac{1}{4\ln 2 + 4})\alpha(k)k^2)
\end{aligned}$$

recalling that $\alpha(k)k^2 \to \infty$ as $k \to \infty$ we note that $\mathbb{P}(\exists U, \vec{a},$ such that $X_{U,\vec{a}}) \to 0$ as $k \to \infty$. Given $\mathbb{P}(\exists U, \vec{a},$ such that $X_{U,\vec{a}}) \to 0$, we conclude there is an integer N such that for all $k \geq N$, $\mathbb{P}(\exists U, \vec{a},$ such that $X_{U,\vec{a}}) < 1$. This implies for all $k \geq N$, there is a $(\alpha(k)k, \alpha(k)k^2)$-comprehensive tournament of order $\lceil (2\alpha(k)\ln 2 + 2)\alpha(k)k^2 2^{\alpha(k)k} \rceil$ as desired. □

Theorem 3.5.3. *Let $\alpha : \mathbb{N} \to (0, 1]$ be a monotone function such that $\alpha(k)k^2 \to \infty$ as $k \to \infty$, and $\alpha(k)k \in \mathbb{N}$ for all k. There exists an integer N depending on α such that for all $k \geq N$, if G is a graph with $\Delta(G) \leq k$ and $d(G) \leq \alpha(k)k$, then $\chi_o(G) \leq \lceil (2\alpha(k)\ln 2 + 2)\alpha(k)k^2 2^{\alpha(k)k} \rceil$.*

Proof. Let $\alpha : \mathbb{N} \to (0, 1]$ be a monotone function such that $\alpha(k)k^2 \to \infty$ as $k \to \infty$, and $\alpha(k)k \in \mathbb{N}$ for all k. By Lemma 3.5.2 there exists a $(\alpha(k)k, \alpha(k)k^2)$-comprehensive tournament T of order $\lceil (2\alpha(k)\ln 2 + 2)\alpha(k)k^2 2^{\alpha(k)k} \rceil$. If G is a graph with $\Delta(G) \leq k$ and $d(G) \leq \alpha(k)k$, then Lemma 3.5.1 implies that G has a homomorphism to T. Thus, $\chi_o(G) \leq \lceil (2\alpha(k)\ln 2 + 2)\alpha(k)k^2 2^{\alpha(k)k} \rceil$ as desired. □

Corollary 3.5.4. *If $\{G_n\}$ is a sequence of graphs satisfying $d(G_n) = o(\Delta(G_n))$, then*

$$\chi_o(G_n) \leq (2 + o(1))\Delta d 2^d$$

where $d = d(G_n)$, and $\Delta = \Delta(G_n)$, and the asymptotics are in Δ.

Notice that in the statement of the corollary given we assume that $d(G_n) = o(\Delta(G_n))$ it is implicit that $\Delta(G_n) \to \infty$. For the sake of any reader less familiar with asymptotic notation, Corollary 3.5.4 is stating that given any graph sequence as required, for all $\varepsilon > 0$ there exists a large enough integer N such that for all $n \geq N$, $\chi_o(G_n) \leq (2 + \varepsilon)\Delta(G_n)d(G_n)2^{d(G_n)}$. In this way, for any sequence or family of graphs where the degeneracy is sublinear in max degree, for any $\varepsilon > 0$, the coefficient from Theorem 3.5.3 may be improved to $2 + \varepsilon$ in the limit as $\Delta \to \infty$.

$d \leq$	$\chi_o(G) \leq$	$\chi_o(G) = O(-)$
$\alpha\Delta$	$\alpha(2\alpha \ln 2 + 2)\Delta^2 2^{\alpha\Delta}$	$O(\Delta^2 2^{\alpha\Delta})$
Δ^α	$(2 + o(1))\Delta^{1+\alpha} 2^{\Delta^\alpha}$	$O(\Delta^{1+\alpha} 2^{\Delta^\alpha})$
$\log_2 \Delta$	$(2 + o(1))\Delta^2 \log_2 \Delta$	$O(\Delta^2 \log \Delta)$
c	$(2c2^c + o(1))\Delta$	$O(\Delta)$

Table 3.3: Asymptotic bounds from Theorem 3.5.3 when $d \ll \Delta$. Note that in this example c is a constant.

Chapter 4

Colouring Graphs of Euler Genus g

4.1 Polynomial Bounds

In this section, we aim to prove the oriented chromatic number of a graph is bounded above by a polynomial in Euler genus. We note that the degree of this polynomial is not our focus, given all prior upper bounds of this form are exponential in Euler genus. Moreover, the strongest conjecture regarding the best upper bound for the oriented chromatic number in Euler genus is by Aravind and Subramanian [8], who conjectured that all graphs of Euler genus g have oriented chromatic number at most $2^{O(\sqrt{g})}$. Thus, any polynomial upper bound proves this conjecture, by demonstrating a vastly better bound. The results in this section appear in the author's paper [18].

The proof of our primary result in this section is more involved than those which appear in Chapter 3. As a result, we require eight lemmas, which vary in complexity, and on their face may not always seem related. As is standard in mathematical writing all lemmas are stated and proven before the theorem that uses them. We note however, that it may be more convenient for the reader to begin with Theorem 4.1.8, then review the lemmas with their applications to the the proof of Theorem 4.1.8 in mind.

We begin by establishing a relationship between the injective chromatic index and the oriented chromatic number of a graph. Let G be a graph. Combining the upper bound $\chi_o(G) \leq \chi_a(G)2^{\chi_a(G)-1}$ from [68] and the upper bound $\chi_a(G) \leq 3^{\chi'_{\text{inj}}(G)}$ from [11], we see that the oriented chromatic number of a graph is bounded above by a double exponential function of the graph's injective chromatic index. The following lemma shows that this upper bound can, in fact, be improved to an exponential function.

Lemma 4.1.1. *For every graph G, if $\chi'_{\text{inj}}(G) = k$, then $\chi_o(G) \leq 4^k$.*

Proof. Let G be a graph with injective chromatic index k, and let $\psi : E(G) \to \{1, \dots, k\}$ be an injective edge coloring of G. Suppose that $E(G)$ has some orientation. We give $V(G)$ an oriented coloring $\phi : V(G) \to 2^{\{1,\dots,k\}} \times 2^{\{1,\dots,k\}}$ by assigning each vertex $v \in V(G)$ the color $\phi(v) = (S_v^+, S_v^-)$, where S_v^+ is the set of colors appearing at arcs outgoing from v, and S_v^- is the set of colors appearing at arcs going into v. We argue that ϕ is an oriented coloring.

First, we argue that ϕ is proper. Indeed, suppose that there exists an arc uv in G such that $\phi(u) = \phi(v)$. Since $\psi(uv) \in S_v^- \cap S_u^+$, the equality $(S_u^+, S_u^-) = (S_v^+, S_v^-)$ implies that $\psi(uv) \in S_v^+ \cap S_u^-$. Then there must exist an arc a_1 going into u of color $\psi(uv)$ as well as an arc a_2 outgoing from v of color $\psi(uv)$, which is a contradiction, as a_1 and a_2 are either at distance 1 or part of a common triangle. Hence, ϕ is proper.

Next, suppose that there exist two arcs uv and $v'u'$ in G so that $\phi(u) = \phi(u')$ and $\phi(v) = \phi(v')$. Since ψ is an injective coloring, it must hold that $S_u^- \cap S_v^+ = \emptyset$. Then, since $(\phi(u), \phi(v)) = (\phi(u'), \phi(v'))$, this implies that $S_{u'}^- \cap S_{v'}^+ = \emptyset$. However, this is a contradiction, since $\psi(v'u') \in S_{u'}^- \cap S_{v'}^+$. Hence, ϕ is an oriented coloring. □

Given this upper bound for the oriented chromatic number in terms of injective chromatic index, we note that it is sufficient to prove that every graph of Euler genus g, is injectively $O(\log(g))$ edge colourable. Unfortunately, this is not true. For instance, Theorem 2.2.3 along with some elementary calculations imply that every complete graph has $\chi'_{\text{inj}}(K_n) = (3 + o(1))g$ where g is the Euler genus of K_n.

Luckily, there are ways to remedy these issues. The rest of the lemmas incrementally make progress towards showing that every graph of Euler genus g has a subgraph containing almost every vertex and almost every edge, which can be injectively $O(\log(g))$ edge coloured. From this point we use Lemma 4.1.1 to colour almost every vertex of a graph G with Euler genus g, and give a unique colour to the remaining vertices of G. Here the phrase 'almost every' is used to avoid some complexity that is inherent to the arguments and statements involved.

Lemma 4.1.2. *Let $k \geq r \geq 2$ be positive integers. There exists a family of $\lceil er^2 \log k \rceil$ subsets $P_i \subseteq \{1, \dots, k\}$ such that for each sequence $(a_1, \dots, a_\ell)$ of $\ell \leq r$ distinct elements from $\{1, \dots, k\}$, there exists a subset P_i such that $a_1 \in P_i$ and $a_2, \dots, a_\ell \notin P_i$.*

Proof. It is enough to prove the lemma under the assumption that $\ell = r$. For $1 \leq i \leq \lceil er^2 \log k \rceil$, we construct a subset P_i by adding each $j \in \{1, \dots, k\}$ to P_i independently with probability $\frac{1}{r}$. We note that the subsets P_i in our family may not all be distinct, but this is not a problem. For a given sequence $(a_1, \ldots, a_r)$, the probability that $a_1 \in P_i$ and $a_2, \ldots, a_r \notin P_i$ is equal to $\frac{1}{r}(1-\frac{1}{r})^{r-1} > \frac{1}{er}$. Therefore, the probability that these conditions do not hold for any subset P_i is less than

$$\left(1 - \frac{1}{er}\right)^{\lceil er^2 \log k \rceil} < \frac{1}{k^r}.$$

As the number of sequences $(a_1, \ldots, a_r)$ with r distinct elements from $\{1, \ldots, k\}$ is less than k^r, the expected number of such sequences that do not satisfy our property for some P_i is less than 1. Thus, with positive probability, there exists a system of subsets P_i that satisfy the lemma. □

Now, using Lemma 4.1.2, we can carry out a deterministic analogue of the random procedure used to prove Theorem 2.7.3 in [18]. This deterministic process is outlined in the proof of the next lemma. As a result, the following lemma can be viewed as a statement of the necessary conditions to implement our strategy, as well as a bound on the resulting number of colours used.

Lemma 4.1.3. *Let G be an oriented graph, and let $X \subseteq V(G)$ be an independent set in G with maximum out-degree d. Let H be a graph on $V(G) \setminus X$ defined so that two vertices $u, v \in V(G) \setminus X$ are adjacent if and only if there exists a vertex $x \in X$ such that $u, v \in N^+(x)$. Then, the set of arcs in G outgoing from X can be partitioned into*

$$\lceil ed^2 \log \chi(H) \rceil (2\Delta^+(G) + 1) = O(d^2 \Delta^+(G) \log \chi(H))$$

star forests which are induced in G.

Proof. Let ϕ be a proper coloring of H with $k = \chi(H)$ colors. Let $\mathcal{P}$ be a set of $\lceil ed^2 \log k \rceil$ subsets $P_i \subseteq \{1, \ldots, k\}$ such that for each sequence $(a_1, \ldots, a_\ell)$ of $\ell \leq d$ distinct elements from $\{1, \ldots, k\}$, there exists a subset P_i such that $a_1 \in P_i$ and $a_2, \ldots, a_\ell \notin P_i$. The set $\mathcal{P}$ exists by Lemma 4.1.2.

Now, we color $N^+(X)$ as follows. For each subset $P_i \in \mathcal{P}$, we execute the following steps.

1. Initialize sets $V_i = \emptyset$, $E_i = \emptyset$.

2. Define $X_i \subseteq X$ as the set of vertices $x \in X$ such that exactly one vertex $v \in N^+(x)$ satisfies $\phi(v) \in P_i$.

3. For each $x \in X_i$, let $z_i(x)$ be the unique out-neighbor of x for which $\phi(z_i(x)) \in P_i$. Update $V_i \leftarrow V_i \cup \{z_i(x)\}$ and $E_i \leftarrow E_i \cup \{xz_i(x)\}$.

4. Define an oriented graph D_i with vertex set V_i and with arcs defined as follows. For any two vertices $u, v \in V_i$, add the arc uv to D_i if and only if either $uv \in E(G)$ or there exists a vertex $x \in X_i$ such that $ux \in E(G)$ and $v = z_i(x)$.

5. Give D_i a proper coloring ψ_i using the set of colors $\{1, \dots, 2\Delta^+(G) + 1\}$.

6. For each arc $xv \in E_i$, color xv with the color $(i, \psi_i(v))$.

Again, if our procedure asks us to color some arc $e \in E(G)$ that has already been colored, then we let the new color of e replace the old color. Note that since each vertex $x \in X_i$ has a unique out-neighbor $z_i(x)$, it follows that for each $v \in V_i$, $\deg^+_{D_i}(v) \leq \deg^+_G(v)$, so D_i has maximum out-degree at most $\Delta^+(G)$. Hence, D_i is a $2\Delta^+(G)$-degenerate graph, and thus Step (5) is possible.

First, we claim that the procedure above colors each arc of $N^+(X)$. Indeed, consider a vertex $x \in X$ and an out-neighbor $u \in N^+(x)$. Write $N^+(x) = \{u, w_1, \dots, w_t\}$, and consider the sequence $S = (\phi(u), \phi(w_1), \dots, \phi(w_t))$. Since $N^+(x)$ induces a clique in H, all elements in S are distinct. Hence, there exists a subset $P_i \in \mathcal{P}$ such that $\phi(u) \in P_i$ and $\phi(w_1), \dots, \phi(w_t) \notin P_i$. Then, by construction, xu is colored with some color $(i, \psi_i(u))$.

Next, we claim that in our coloring of $N^+(X)$, each color class is a star forest which is induced in G. Indeed, consider two arcs xu and yv in $E(G)$ at distance 1 in G, where $x, y \in X$ and $u, v \in V(G) \setminus X$. Suppose that both xu and yv are colored with the color (i, j). If this occurs, then it must hold that $u = z_i(x)$, $v = z_i(y)$, and $u, v \in V_i$. Since xu and yv are at distance 1 in G, and since X is an independent set, one of the following cases must hold without loss of generality.

1. $yu \in E(G)$. Since $v = z_i(y)$, $\phi(v) \in P_i$ and $\phi(u) \notin P_i$. Then $u \notin V_i$, a contradiction.

2. $uy \in E(G)$. As $v = z_i(y)$, uv is an arc of D_i. Hence, $\psi_i(u) \neq \psi_i(v)$, and xu and yv cannot both be colored with (i, j), a contradiction.

3. $uv \in E(G)$. Then, as before, uv is an arc of D_i, and $\psi_i(u) \neq \psi_i(v)$, which again gives a contradiction.

Therefore, each color class in $N^+(X)$ produced by our procedure is an induced star forest in G. Since each color class is of the form (i, j), where $i \in \{1, \dots, \lceil ed^2 \log k \rceil\}$ and $j \in \{1, \dots, 2\Delta^+(G) + 1\}$, we use at most $\lceil ed^2 \log k \rceil (2\Delta^+(G) + 1)$ colors in our coloring. This completes the proof. $\square$

While Lemma 4.1.3 gives us a tool for partitioning the edges of an oriented graph G into induced star forests, it is unclear how to estimate the chromatic number of the graph H defined in the lemma. In the following lemmas, we will show that if G is an oriented graph of Euler genus g, then whenever we apply Lemma 4.1.3 to an independent set $X \subseteq V(G)$, we can bound the value $\chi(H)$ by a function of g and $\Delta^+(G)$.

Given a hypergraph $\mathcal{H}$, the *Levi graph* of $\mathcal{H}$ is the bipartite graph L with vertex set $V(\mathcal{H}) \cup E(\mathcal{H})$ such that for each $v \in V(\mathcal{H})$ and $e \in E(\mathcal{H})$, $ve \in E(L)$ if and only if $v \in e$. Adopting a standard convention (see e.g. [39]), we say that $\mathcal{H}$ has Euler genus g if and only if the Levi graph of $\mathcal{H}$ has Euler genus g. Additionally, we define the *clique graph* $K(\mathcal{H})$ of $\mathcal{H}$ as the graph on $V(\mathcal{H})$ such that two vertices $u, v \in V(\mathcal{H})$ are adjacent in $K(\mathcal{H})$ if and only if u and v belong to a common edge of $\mathcal{H}$. Observe that $K(\mathcal{H})$ is formed by replacing each edge of $\mathcal{H}$ with a clique. The notion of a clique graph will be useful to us, as the graph H in the statement of Lemma 4.1.3 can be defined as the clique graph of a certain hypergraph.

In the following lemmas, we establish an upper bound for the chromatic number of a clique graph obtained from a hypergraph of Euler genus g.

Lemma 4.1.4. *If $\mathcal{H}$ is a hypergraph with edges of size at most r and Euler genus at most $g \geq 2$, then $K(\mathcal{H})$ has a vertex of degree at most $20r^2\sqrt{g} - 1$.*

Proof. For our proof, we assume that each edge in $\mathcal{H}$ contains at least two vertices, as edges of size one have no influence on $K(\mathcal{H})$. We write L for the Levi graph of $\mathcal{H}$. We partition $E(\mathcal{H})$ into parts E_2 and $E_{\geq 3}$, where E_2 consists of all edges in $E(\mathcal{H})$ of size 2, and $E_{\geq 3}$ contains all other edges of $\mathcal{H}$. We aim to find upper bounds for E_2 and $E_{\geq 3}$.

First, we consider the edge set E_2. The graph $H_2 = (V(\mathcal{H}), E_2)$ is a topological minor of L, so the Euler genus of H_2 is at most g. Therefore, by Euler's formula, $|E_2| \leq 3|V(\mathcal{H})| + 3g - 6$.

Next, we consider the edge set $E_{\geq 3}$. Let $L_{\geq 3}$ be the Levi graph of the hypergraph $(V(\mathcal{H}), E_{\geq 3})$, and consider an embedding of $L_{\geq 3}$ on a surface of minimum Euler genus. Since $L_{\geq 3}$ is a subgraph of L, the Euler genus of $L_{\geq 3}$ is at most g. Thus, by Euler's formula,

$$\begin{aligned}
|V(L_{\geq 3})| - |E(L_{\geq 3})| + |F(L_{\geq 3})| &\geq 2 - g \\
|V(L_{\geq 3})| - \frac{1}{2}|E(L_{\geq 3})| &\geq 2 - g \\
|V(\mathcal{H})| + |E_{\geq 3}| - \frac{3}{2}|E_{\geq 3}| &\geq 2 - g \\
|E_{\geq 3}| &< 2|V(\mathcal{H})| + 2g
\end{aligned}$$

Now, let $u_1, \ldots, u_{|E(\mathcal{H})|}$ be the vertices in L corresponding to the edges of $\mathcal{H}$. We observe that

$$\begin{aligned}
|E(K(\mathcal{H}))| &\leq \binom{\deg u_1}{2} + \binom{\deg u_2}{2} + \cdots + \binom{\deg u_{|E(L)|}}{2} \\
&< (\deg u_1)^2 + (\deg u_2)^2 + \cdots + (\deg u_{|E(\mathcal{H})|})^2 \\
&\leq r^2|E(\mathcal{H})| = r^2(|E_2| + |E_{\geq 3}|) \\
&< r^2(5|V(\mathcal{H})| + 5g).
\end{aligned}$$

Now, if $|V(\mathcal{H})| < \sqrt{g}$, then $K(\mathcal{H})$ clearly has a vertex of degree at most $20r^2\sqrt{g} - 1$. Otherwise, $|V(\mathcal{H})| \geq \sqrt{g}$, and $K(\mathcal{H})$ has a vertex of degree at most

$$2|E(K(\mathcal{H}))|/|V(\mathcal{H})| < 2r^2(5 + 5\sqrt{g}) < 20r^2\sqrt{g} - 1.$$

This completes the proof. □

Lemma 4.1.4 gives us the following corollary.

Lemma 4.1.5. *If $\mathcal{H}$ is a hypergraph with edges of size at most r and Euler genus at most $g \geq 2$, then $\chi(K(\mathcal{H})) \leq 20r^2\sqrt{g}$.*

Proof. Suppose the lemma is false, and let $\mathcal{H}$ be the hypergraph on the fewest number of vertices for which the lemma does not hold. By Lemma 4.1.4, $K(\mathcal{H})$ has a vertex u of degree at most $20r^2\sqrt{g} - 1$. Consider the hypergraph $\mathcal{H}'$ on $V(\mathcal{H}) \setminus \{u\}$ with edge set $\{e \setminus \{u\} : e \in E(\mathcal{H})\}$. If we write L for the Levi graph of $\mathcal{H}$ and L' for the Levi graph of $\mathcal{H}'$, clearly L' is a subgraph of L, so L' has genus at most g. Hence, as $\mathcal{H}$ is a minimum counterexample, $K(\mathcal{H}')$ has a proper coloring with $20r^2\sqrt{g}$ colors. Furthermore, it is easy to check that $K(\mathcal{H}) \setminus \{u\} = K(\mathcal{H}')$. Therefore, we may properly color $K(\mathcal{H}) \setminus \{u\}$ with $20r^2\sqrt{g}$ colors, and since u has at most $20r^2\sqrt{g} - 1$ neighbors, we may extend this coloring to u. Hence $\mathcal{H}$ is in fact not a counterexample, giving a contradiction and completing the proof. □

We have now finished proving all the necessary tools to injectively edge colour graphs of Euler genus g. Notice that at this point we still cannot prove anything approaching a polynomial upper bound for the oriented chromatic number in terms of Euler genus g. This is primarily because both Lemma 4.1.3 and Lemma 4.1.5 involve a term related to the maximum degree. That is, Lemma 4.1.3 implicitly uses Δ^+, the maximum out-degree, while Lemma 4.1.5 uses r, the maximum size of a hyperedge in $\mathcal{H}$. Observe that if we consider H from Lemma 4.1.3 as a clique graph of a hypergraph $\mathcal{H}$, then these two terms are related. But Δ^+ and r need not be bounded.

Our argument requires us to colour a large subgraph of G, precisely so that when applying Lemma 4.1.3 and Lemma 4.1.5 these parameters are bounded. Also note that Δ^+ is a parameter of a directed graph, while injective edge colouring does not depend on orientation. Hence, given two orientations G' and G'' of the same simple graph G, when injective edge colouring G we can choose to G' or G'', so that Δ^+ is minimised. Of course the same is not true when oriented colouring. However, Lemma 4.1.1 allows us to take advantage of this ability to switch orientation when building oriented colourings from injective edge colourings.

The next lemma is not difficult to find, or to prove, but turns out to be surprisingly useful. In effect, we prove that if a graph G of bounded Euler genus has large order, then G has many low degree vertices. Moreover, there exists an ordering of the vertices of any such graph, where almost every vertex has at most 6 back neighbours.

Lemma 4.1.6. *Let $k \geq 1$ be an integer. If G is a graph of Euler genus at most $g \geq 2$ and minimum degree at least $k+6$, then G has fewer than $\frac{6g}{k}$ vertices.*

Proof. By Euler's formula, $|V(G)| - \frac{1}{3}|E(G)| > -g$. Rearranging this,

$$\sum_{v \in V(G)} (\deg(v) - 6) < 6g.$$

If each vertex has degree at least $k+6$, then the number of terms in this sum is less than $\frac{6g}{k}$, completing the proof. □

We need one more lemma about the oriented chromatic number before proving our main result.

Lemma 4.1.7. *Let G be a graph, and let $U \subseteq V(G)$. Then $\chi_o(G) \leq |U| + \chi_o(G \setminus E(G[U]))$.*

Proof. Consider a fixed orientation of $E(G)$. We give G a proper oriented coloring as follows. First, we define an oriented coloring ϕ of $G \setminus E(G[U])$ that uses $\chi_o(G \setminus E(G[U]))$ colors. Then, we define a new coloring ψ by recoloring each vertex of U with a new unique color. We show that ψ is an oriented coloring of G.

We first claim that ψ is a proper coloring. Indeed, suppose that there exist two adjacent vertices $u, v \in V(G)$ so that $\psi(u) = \psi(v)$. Since ϕ is a proper coloring, it must follow without loss of generality that $u \in U$. However, then u is the only vertex with the color $\psi(u)$, so $\psi(u) \neq \psi(v)$, a contradiction.

Next, suppose that there exist arcs uv and $v'u'$ in G so that $(\psi(u), \psi(v)) = (\psi(u'), \psi(v'))$. Since ϕ is an oriented coloring, it follows that one of u, v, u', v' belongs to U. If $u \in U$, then since u is the only vertex with color $\psi(u)$, it follows that $u = u'$. If $v = v'$, then G contains a digon, a contradiction. Therefore, since $\psi(v) = \psi(v')$, it follows that $v, v' \notin U$. Hence, $(\psi(u), \psi(v)) = (\psi(u'), \psi(v'))$, contradicts either assumption that ϕ is an oriented coloring or that G contains no digon. Therefore, ψ is an oriented coloring. □

We are ready to prove our upper bound, that if G is a graph with sufficiently large Euler genus g then $\chi_o(G) \leq g^{6400}$. This polynomial upper bound in g gives a proof of Conjecture 2.5.5, as it greatly improves the conjectured bound. We can prove a slightly better upper bound however this improvement is not significant and leads to a more complicated expression. As we do not believe either bound is remotely close to being tight, we state the weaker bound due to its preferable aesthetics.

Theorem 4.1.8. *If G is a graph of sufficiently large Euler genus g, then*

$$\chi_o(G) \leq g^{6400}.$$

Proof. We let G be a graph of Euler genus g, and we assume that g is sufficiently large. Rather than bounding $\chi_o(G)$ by considering an explicit orientation of $E(G)$, we will bound $\chi_o(G)$ by estimating the injective chromatic index of a certain subgraph of G and then using Lemmas 4.1.1 and 4.1.7. Suppose then that G is undirected.

We write $n = |V(G)|$, and we order $V(G)$ as follows. We iterate through $i = n, n-1, \dots, 3, 2, 1$, and on each iteration we let v_i be the vertex of minimum degree in $G \setminus \{v_{i+1}, \dots, v_n\}$. We then give G a proper coloring ϕ by iterating through $i = 1, \dots, n$ and coloring v_i with the least available positive integer which has not already been used at a neighbor. Next, we partition $V(G)$ into parts $V_1 = \{v_1, \dots, v_{6g}\}$ and $V_2 = \{v_{6g+1}, \dots, v_n\}$. For each vertex $v_i \in V_2$, v_i is the minimum-degree vertex in $G[v_1, \dots, v_i]$; hence, by Lemma 4.1.6, for each value $i > 6g$, the vertex v_i has at most 6 neighbors v_j for which $j < i$. Therefore, for each vertex $v_i \in V_2$, $\phi(v_i) \leq 7$. We also orient $E(G)$ so that each edge $v_i v_j$ is oriented from v_i to v_j if and only if $i > j$. Note that under this orientation, each vertex $v_i \in V_2$ has out-degree at most 6.

Now, we define $G' = G \setminus E(G[V_1])$, and we aim to bound $\chi'_{\text{inj}}(G')$. For each color $c \in \{1, \dots, 7\}$, let $X_c \subseteq V_2$ be the independent set consisting of those vertices in V_2 of color c. We will apply Lemma 4.1.3 to partition $N^+(X_c)$ into induced star forests. We write $\mathcal{H}_c$ for the hypergraph on $V(G') \setminus X_c$ with the edge set $\{N^+(x) : x \in X_c\}$, and we write $k = \chi(K(\mathcal{H}_c))$. By Lemma 4.1.5, $k \leq 20 \cdot 6^2 \sqrt{g}$. Since G' is 6-degenerate and has maximum out-degree 6, Lemma 4.1.3 tells us that $N^+(X_c)$ can be partitioned into $13\lceil 36e \log k \rceil \leq (234e + o(1)) \log g$ star forests which are induced in G'. By repeating this process for all 7 color classes of G', we find an injective edge-coloring of G' using at most $(1638e + o(1)) \log g$ colors.

Finally, by Lemma 4.1.1, $\chi_o(G') \leq 4^{(1638e+o(1)) \log g} < g^{6400} - 6g$ for large g. Since $G' = G \setminus E(G[V_1])$, it then follows from Lemma 4.1.7 that $\chi_o(G) \leq \chi_o(G') + |V_1| \leq g^{6400}$, completing the proof. $\square$

We conclude this section with a randomized construction which shows the existence of oriented graphs with large Euler genus g and oriented chromatic number at least $g^{\frac{2}{3} - o(1)}$. Rather than directly estimating the oriented chromatic number of a randomized construction G, we instead consider its 2-dipath chromatic number $\chi_2(G)$. The randomized construction that we use is a standard method for constructing graphs for which various coloring parameters are large, such as acyclic chromatic number [2, 3], star chromatic number [31], and non-repetitive chromatic number [1]. This construction shows us that the exponent 6400 in Theorem 4.1.8 is correct within a factor of less than 10000.

Proposition 4.1.9. *There exists a constant $c > 0$ such that for each value $g \geq 2$, there exists an oriented graph G of Euler genus g for which $\chi_o(G) \geq \chi_2(G) \geq c \left(\frac{g^2}{\log g}\right)^{1/3}$.*

Proof. We may assume that g is sufficiently large, as the statement holds for small values of g by letting c be sufficiently small. We set $p = \sqrt{\frac{150 \log n}{n}}$, and we choose n to be as large as possible so that n is even and $pn^2 \leq g$.

We let G be an oriented graph on n vertices which is constructed as follows. For each pair of vertices u and v, we join u and v by an edge e independently with probability p, and if e is added to G, we give e one of the two possible orientations uniformly at random. By a Chernoff bound (see e.g. [57, Chapter 4]), it holds a.a.s. that $|E(G)| < pn^2$, and hence G a.a.s. has Euler genus less than g.

We aim to show that a.a.s., $\chi_2(G) > n/2$. To this end, we consider a fixed coloring ϕ of $V(G)$ with $n/2$ colors, and we aim to estimate the probability that ϕ is a proper 2-dipath coloring of G. We obtain a subgraph G' of G by deleting at most one vertex from each color class of ϕ so that each color class of G' has an even number of vertices. Clearly, $|V(G')| \geq n/2$. We partition each color class of G' into vertex sets of size 2, which gives a partition Π of $V(G')$ in which each part $P \in \Pi$ consists of exactly two vertices which have the same color. We consider two distinct parts $P = \{u, v\}$ and $P' = \{u', v'\}$ in Π. We observe that if G' contains the arcs uu' and $u'v$, then ϕ is not a 2-dipath coloring of G. The probability that G' contains both arcs uu' and $u'v$ is $p^2/4$, and the number of ways to choose two distinct parts $P, P' \in \Pi$ is at least $\binom{\lceil n/4 \rceil}{2} > \frac{1}{36}n^2$. Therefore, the probability that ϕ is a proper 2-dipath coloring of G is at most

$$(1 - p^2/4)^{\frac{1}{36}n^2} < \exp\left(-\frac{1}{144}(pn)^2\right).$$

Therefore, by a union bound, the probability that G has a 2-dipath coloring is less than

$$n^n \exp\left(-\frac{1}{144}(pn)^2\right) = \exp\left(n \log n - \frac{1}{144}(pn)^2\right) = o(1).$$

Hence, G a.a.s. has no proper 2-dipath coloring using $n/2$ colors. Therefore, a.a.s.,

$$\chi_2(G) > n/2 = \frac{1}{2}\left(\frac{pn^2}{\sqrt{150 \log n}}\right)^{2/3} = \Omega\left(\left(\frac{g^2}{\log g}\right)^{1/3}\right).$$

Finally, we may increase the Euler genus of G to exactly g without decreasing its 2-dipath chromatic number by adding sufficiently many disjoint copies of K_5, completing the proof. □

4.2 Lowering the Order of the χ_o Upper Bound

Having established that the oriented chromatic number of a graph G is bounded by a polynomial function of its Euler genus g, we turn our attention toward reducing the degree of this polynomial. The results in this section appear in the author's paper [18]. While we are unable to substantially improve the bound of $\chi_o(G) \leq g^{6400}$ given in Theorem 4.1.8, we show that in order to improve the exponent of 6400, it is sufficient to establish an improved upper bound for $\chi_2(G)$. Unlike the oriented coloring problem, which has global constraints, the constraints of the 2-dipath coloring problem are entirely local, which often makes $\chi_2(G)$ much easier to estimate than $\chi_o(G)$. With this in mind, our main goal in this section is to prove Theorem 4.2.3, which shows that an upper bound on $\chi_2(G)$ in terms of g implies a similar upper bound on $\chi_o(G)$.

Lemma 4.2.1. *For each value $d \geq 2$ and $k \geq 5$, there exists a $(k, d, \lceil 8^d \log k \rceil)$-full graph.*

Proof. We let $N = \lceil 8^d \log k \rceil$. We let H be a random orientation of the complete k-partite graph $K_{N,\dots,N}$. We consider a fixed value $i \in \{1, \dots, k\}$ and a fixed ordered subset $U = (u_1, \dots, u_d) \subseteq \bigcup_{j \neq i} P_j$, as well as a fixed vector $q \in \{-1, 1\}^d$. The probability that a given vertex $x \in P_i$ satisfies $F(U, x, G) = q$ is 2^{-d}, so the probability that no vertex $v \in P_i$ satisfies $F(U, x, G) = q$ is at most $(1 - 2^{-d})^N < \exp(-2^{-d} N)$. Therefore, taking a union bound over all possible values $i \in [k]$, all ordered subsets $U \subseteq \bigcup_{j \neq i} P_j$ of size d, and all vectors $q \in \{-1, 1\}^d$, the probability p that H is not (k, d, N)-full satisfies

$$p \leq k \cdot (kN)^d 2^d \exp(-2^{-d} N).$$

The rest of the proof aims to show that $p < 1$. We observe that

$$\begin{aligned}
\log p &< (d+1)(\log k + \log N + \log 2) - \frac{N}{2^d} \\
&= (d+1)(\log k + \log \lceil 8^d \log k \rceil + \log 2) - \frac{\lceil 8^d \log k \rceil}{2^d} \\
&< (d+1)(2 \log k + \log 8^d + \log 2) - 4^d \log k \\
&= (d+1) \left((2 - \frac{4^d}{d+1}) \log k + (3d+1) \log 2 \right)
\end{aligned}$$

The $\log k$ term in the last expression has a negative coefficient for all $d \geq 2$, and therefore this expression is decreasing with respect to k. Hence,

$$\log p < (d+1)\left((2 - \frac{4^d}{d+1})\log 5 + (3d+1)\log 2\right),$$

which is negative for all $d \geq 2$. Therefore, $p < 1$, and thus with positive probability, the oriented graph H which we have constructed is $(k, d, \lceil 8^d \log k \rceil)$-full. $\square$

Using the same ideas as from Chapter 3.2 this implies the following bound on the oriented chromatic number of graphs with bounded 2-dipath chromatic number and degeneracy.

Lemma 4.2.2. *Let $d \geq 2$ and $k \geq 5$. If G is a d-degenerate graph for which $\chi_2(G) = k$, then $\chi_o(G) \leq k\lceil 8^d \log k \rceil$.*

Proof. Let G is a d-degenerate graph for which $\chi_2(G) = k$. By Lemma 4.2.1, there exists a $(k, d, \lceil 8^d \log k \rceil)$-full graph H. By Lemma 3.2.1, G has an oriented homomorphism to H. This concludes the proof. $\square$

Now, we are ready to prove Theorem 4.2.3.

Theorem 4.2.3. *There exists a constant C such that if G is a graph of Euler genus $g \geq 0$ satisfying $\chi_2(G) = k$, then*

$$\chi_o(G) < C(k \log k + g + 1).$$

Proof. We will show that the constant $C = 2^{20}$ is sufficiently large. We write $n = |V(G)|$. If $n \leq 6g$, then $\chi_o(G) \leq 6g$. If $g \leq 1$, then we may write $a = \chi_a(G)$ and use the inequalities $a \leq 7$ [3] and $\chi_o(G) \leq a2^{a-1} \leq 448$ [68] to finish the proof.

Otherwise, we assume that $g \geq 2$ and $n > 6g$. Let $v_1, \ldots, v_n$ be an ordering of the vertices of G, so that for each $i \in [n]$, v_i is has minimum degree in the graph $G[\{v_1, \ldots, v_i\}]$. We write $V_1 = \{v_1, \ldots, v_{6g}\}$ and $V_2 = \{v_{6g+1}, \ldots, v_n\}$. We define $G' = G \setminus E(G[V_1])$, and as in the proof of Theorem 4.1.8, G' is a 6-degenerate graph.

If $\chi_2(G') < 5$, then $\chi_o(G') \leq 8$ by the inequality $\chi_o(G') \leq 2^{\chi_2(G')-1}$ [49]. Otherwise, by Lemma 4.2.2, $\chi_o(G') < \chi_2(G')\lceil 8^6 \log \chi_2(G') \rceil \leq k\lceil 8^6 \log k \rceil < Ck \log k$. In both cases, by Lemma 4.1.7,

$$\chi_o(G) \leq |V_1| + \chi_o(G') < C(k \log k + g + 1).$$

This concludes the proof. □

Chapter 5

Future Work

In this thesis we study the vertex colourings, with a focus on oriented colourings, of families of graphs with bounded maximum degree, or families of graphs with bounded degeneracy. At times, such as Chapter 3.3 and Chapter 3.4, we consider families of graphs with maximum degree at most k, which implies a degeneracy is at most k. We also consider families with bounded degeneracy, which may have arbitrarily large maximum degree, such as in Chapter 3.2 and all of Chapter 4. To complete this picture, we consider graphs with maximum degree at most k, and degeneracy as most $\alpha(k)$, where $\alpha(k) \leq O(k)$ is an arbitrary function of k, in Chapter 3.1 and Chapter 3.5.

The novel results in this thesis appear in [18] and [21], both of which are papers by the author and his collaborators. The major contributions of this work is a sequence of upper bounds on the oriented chromatic number, which are presented in Chapter 3 and Chapter 4 of this thesis. As a result, the future work discussed in this conclusion will focus on questions relating to how close the bounds appearing in Chapter 3 and Chapter 4 are to being tight. Attention is also paid to open problems which arise from the methods used in Chapter 3 and Chapter 4.

We begin by asking if the upper bound from Theorem 3.1.1 on the 2-dipath chromatic number of a graph with maximum degree at most k is asymptotically tight up to a a constant factor? Notice here that we do not assume degeneracy is bounded to be less than k, hence we are considering graphs with maximum degree at most k and any minimum degree less than or equal to k. To the author's knowledge, the bound from Theorem 3.1.1 is the first attempt to bound the 2-dipath chromatic number of graphs with degree at most k, for an arbitrarily large k, appearing in the literature. The proof of Theorem 3.1.1 is not challenging to obtain, and proceeds by a classical method, that is a greedy colouring strategy, which is known to often produce bounds that are far from tight for other kinds of vertex colouring. Furthermore, despite some effort, the author was unable to produce a graph with maximum degree k whose 2-dipath chromatic number is larger than $\Omega(k \log(k))$. Hence, we conjecture that the bound in Theorem 3.1.1 is not asymptotically tight.

Conjecture 5.0.1. *If G is a graph with maximum degree at most Δ, then $\chi_2(G) = o(\Delta^2)$.*

However, Theorem 3.1.1 provides a bound in terms of maximum degree and degeneracy. So a natural question to ask is whether the bound in Theorem 3.1.1 becomes tight if we add assumptions about the value of the degeneracy of our graph. We conjecture that under correct assumptions about degeneracy, the bound from Theorem 3.1.1 is in fact tight.

Conjecture 5.0.2. *There exists a family of graphs $\mathcal{G}$ with bounded degeneracy with the following property. There exists a constant $c > 0$ such that for all integers k, there exists $G_k \in \mathcal{G}$ such that $\Delta(G_k) \leq k$ and $\chi_2(G_k) \geq ck$.*

We note that Theorem 3.1.1 implies that if Conjecture 5.0.2 is true, then it is best possible. Furthermore, if Conjecture 5.0.2 is true, then Theorem 3.5.3 implies that the oriented chromatic number and 2-dipath chromatic number of those graphs G_k differ by at most a constant factor. Note that such a constant factor will depend on the upper bound for degeneracy in the family $\mathcal{G}$.

Next, we conjecture that the lower bound from Theorem 3.4.2 on the best upper bound for the oriented chromatic number of graphs with maximum degree k, can be improved. Notice that the best possible result of this kind would be that there exists a graph G with maximum degree at most k, such that the oriented chromatic number of G is at least $\Omega(k^2 2^k)$. Of course, all the bound we discuss here are asymptotic, so we assume k tends to infinity.

Conjecture 5.0.3. *There exists a constant $c > 0$ such that for all integers k, there exists a graph with maximum degree at most k and $\chi_o(G) \geq c2^k$.*

Next, we address our bound from Chapter 4. In particular, the bounds from Theorem 4.1.8 and Proposition 4.1.9. We begin by conjecturing that the oriented chromatic number is at most linear in Euler genus.

Conjecture 5.0.4. *There exists a constant c such that if G is a graph with Euler genus $g \geq 1$, then $\chi_o(G) \leq cg$.*

While there is not an abundance of evidence for this upper bound appearing in this thesis, the author believes it to be correct, at least up to a sub-polynomial factor. We do not explore why here, as this pertains to yet unpublished work. Moreover, the author believes that again, at least up to a sub-polynomial factor, a linear upper bound is best possible. Hence, the following conjecture.

Conjecture 5.0.5. *There exists a constant $C > 0$ such that if $g \geq 1$ be an integer, then there exists a graph G with Euler genus at most g, where $\chi_o(G) \geq Cg$.*

This concludes the thesis.

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